TOPICS IN PHILOSOPHICAL LOGIC

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I take pleasure in dedicating this book to my students who have over the years been also my collaborators in logical research, and in particular to:

BAS VAN FRAASSEN

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ERNEST SOSA

PREFACE

The aim of the book is to introduce the reader to some new areas of logic which have yet to find their way into the bulk of modern logic books written from the more orthodox direction of the mainstream of developments. Such a work seems to me much needed, both because of the intrinsic value and increasing prominence of the nonstandard sector of logic, and because this particular sector is of the greatest interest from the standpoint of philosophical implications and applications.

This book unites a series of studies in philosophical logic, drawing for the most part on material which I have contributed to the journal literature of the subject over the past ten years. Despite the fact that some of these essays have been published in various journals at different times, they possess a high degree of thematic and methodological unity. All of these studies deal with material of substantial current interest in philosophical logic and embody a fusion of the modern techniques of logical and linguistic-philosophical analysis for the exploration of areas of logic that are of substantial philosophical relevance.

Chap. VII on 'Venn Diagrams for Plurative Syllogisms' was written in collaboration with my student Mr. Neil A. Gallagher, and Chap. XIII on 'Topological Logic' was written in collaboration with my student Mr. James W. Garson. I am grateful to these gentlemen for agreeing to the inclusion of this material here. I express my appreciative thanks to Miss Dorothy Henle and Miss Judy Bazy for their help in preparing this material for the printer and in seeing it through the press. I should also like to thank Mr. Alasdair Urquhart for his help with the proofreading.

Pittsburgh, August, 1967

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CHAPTER I

RECENT DEVELOPMENTS IN PHILOSOPHICAL LOGIC

1. INTRODUCTION

The mainstream of the development of modern logic since the pioneering days from Boole to Frege has moved very decidedly in the direction of *mathematical* interests and applications. And, in fact, mathematics continues to the present day to occupy a central position on the logical stage. This may be illustrated – among many other ways – by the current prominence of what might be characterized as 'the arithmetical sector' of logic, including algorithmic theory, recursive functions, the calculi of lambda conversion, the logical theory of computability and of effective processes generally, among other components of lesser renown. The computer, and the whole host of technical issues that revolve about it, have had an enormous and reciprocally stimulative impact on recent work in logic. Results of great importance and interest continue to be obtained in this mathematical sector of logic, witness Paul J. Cohen's remarkable proof of the independence of the continuum hypothesis.

However, the continuing of this long-standing mathematical tendency has masked and obscured a highly significant cluster of developments in logic of a more recent vintage. The eventuation to which I allude is the phenomenal recent spurt of growth of logic in directions bearing on philosophical considerations. The last ten or fifteen years especially – though there were, to be sure, earlier stirrings – have seen the flourishing and accelerating growth of branches of logical theory developed specifically with such philosophical applications in mind. It is also worth noting that there is also a growing interest in the 'logic' of natural languages, particularly in the evaluation of the validity of reasoning conducted in such languages, rather than in the more formalized systems used, e.g., in mathematics. This has come about largely under the impetus of the 'ordinary language' school of philosophy.

The principal objective of the present chapter is to give a brief but synoptic survey of this important phase of the ongoing history of logic.

Moreover, I propose to offer some observations regarding the significance of these developments, and to give some consideration to the prospects that augur for the future.

2. A NOTABLE FEATURE OF THE CURRENT SITUATION IN LOGIC

In Appendix A an attempt has been made to construct a map of the terrain of logic as it appears at the present writing. In this enterprise we have not concerned ourselves with matters of detail or with the minutiae of alternative approaches, but have endeavored to give a somewhat gross overview of the 'big picture'.

The material of Appendix A can for the most part safely be left to speak for itself. However, one particular feature of the map will here be singled out for explicit consideration and discussion. I advert to the size, scope, richness, and diversity of category E ('Philosophical Developments'). This phenomenon is so striking as to warrant explicit remark all of itself. Moreover, material of substantially philosophical bearing and interest is by no means confined to this category. For virtually the whole of the subcategories A3 ('Unorthodox Modern Logic') and B ('Metalogic') cannot but also be regarded in this light, being of preeminently philosophical bearing. A very sizeable sector of current logic is thus clearly oriented in specifically philosophical directions. This fact is all the more striking when one considers it in an historico-bibliographical perspective.

In Appendix B we have given a concise and selective bibliography of recent literature of philosophical logic. In many or most cases, the works that have been listed are not only significant expository sources, but actually pioneering contributors to the specific topic at issue. This feature serves to bring out in a very forceful and striking way the *recentness* of the cultivation of the philosophical reaches of modern logic. The great bulk of work in this area has appeared in the last decade. The bibliography provides a clear indication not only of the lively activity on this particular sector of the subject, but also of the element of newness that is present here. The majority of its entries (41 of 68) represent publications of the 1960's.

3. PROSPECTS AND PORTENTS

We have noted as a significant recent tendency in the development of modern logic the extensive and energetic cultivation of philosophically oriented branches of the subject. In general terms, the prospect for the future seems clear. There is little if any room for doubt that this tendency will not only continue, but intensify and develop in the years ahead. I should like to offer a few observations as to what this means for philosophy, for logic, and for the relationship between the two.

For philosophy, the intensified cultivation of philosophical logic means, first of all, the creation of a tremendously valuable *opportunity*. With respect to a certain not insignificant class of philosophical problems, the instrumentalities are now in hand for dealing with the relevant issues in an exact, precise, incontrovertible, and essentially decisive manner. Beginning in the area of epistemology and ontology, this tendency to the precise and formally exact treatment of philosophically relevant problems has recently made its way into other areas: especially in the area of ethical and normative concepts (deontic logic, preference logic, the logic of action). In certain sectors of the subject, there is now a genuine prospect of a continuing, cumulative, and collaborative progress – of the sort that philosophy has long envied the sciences. This trend – which one cannot but regard as now established beyond retrogression – may be viewed as perhaps the major permanent heritage of logical positivism in promoting and popularizing the philosophical application of logical technique.

It should be stressed, on the other hand, that the phenomenon which we have just cast in the role of a valuable opportunity also has certain significant inherent dangers. The existence of a method of investigation that holds good promise of success in a given area of a subject exercises a potent magnetic influence in attracting attention and effort to this sector. In consequence a danger arises that attention may be diverted from those issues – generally of no less and frequently of much greater intrinsic interest and importance – that are not amenable to resolution by the instrumentalities and techniques at issue. Significant progress in the subject as a whole may thus come to be sacrificed in the course of securing minor achievements in some of its branches: workers may be diverted from fruitful efforts in the less tidy areas of philosophy only to deploy logico-analytical virtuosity on substantively trivial issues.

During the 1930's and early 1940's, a thoughtful observer might well have tended towards the conclusion that logic would break off from the ancient moorings that kept it joined to philosophy, and either link itself to mathematics, or (more probably) go its own way as an independent discipline. This development would have seemed only natural against the backdrop of the long series of special sciences which, following the lead of *philosophia naturalis*, cut themselves off from philosophy to set up as special sciences in their own right. It has by now become plain as a pikestaff that this – from the angle of philosophy much to be regretted – development will not come to pass. The phenomenon upon which our attention has here been centered, the extensive and intensive development of branches of logic of specifically philosophical applicability, will of itself assure a continuing close connection between these disciplines.

Finally we must consider the matter from the standpoint of logic itself. Here, alas, the outlook is not so unmixedly favorable as one might wish for. There is, I am convinced, nothing for it but that in the fullness of time there will increasingly come to be a fission in the subject. There will come to be an increasingly wide gap between mathematical and philosophical logic, a gulf only occasionally bridged over by a rare mind of more than ordinary capacity and versatility. I am firmly persuaded that this development, which cannot but be viewed as intrinsically unfortunate, is, in effect, inevitable in the long run. Its impact cannot be prevented: the most that can be done is to cushion against consequences of the most dire sort. Its seriousness can, I believe, be mitigated only by a resolute determination on the part of those responsible for the training of specialists in logic in departments of philosophy and of mathematics to insist that students being trained on either side of the divide also attain a thorough familiarity with the way in which things are done on the other side.

4. CONCLUSION

Our brief survey of the structure of modern logic has brought into clear focus a current trend of substantial significance: the increasingly flourishing growth of the philosophically oriented sector of logic in the recent past. We have scrutinized this trend and have endeavored to assess its significance for logic, for philosophy, and for the relationship between them. We are led inescapably to the view that – certain inherent liabilities

notwithstanding – this notable development is greatly to be welcomed from every point of view, excepting perhaps one alone, namely that of logic viewed as a unified discipline exhibiting, across the whole of its great extent, a tight integrative cohesion.[1]

[1] This chapter is an expanded version of a paper published (under the same title) in *Logique et Analyse* **9** (1966) 269–279.

APPENDIX A

A MAP OF LOGIC

A. *Basic Logic*

1. Traditional Logic
   a. Aristotelian logic
      i. theory of categorical propositions
      ii. immediate inference
      iii. syllogistic logic
   b. Other developments
      i. the medieval theory of *consequentiae*
      ii. discussions of the 'laws of thought' in idealistic logic
2. Orthodox Modern Logic
   a. propositional logic
   b. quantificational logic
      i. predicate logic
      ii. logic of relations
3. Unorthodox Modern Logic
   a. modal logic
      i. alethic modalities
      ii. physical modalities (see D1b)
      iii. deontic modalities (see E1b)
      iv. epistemic modalities (see E3b)
   b. many-valued logic
   c. nonstandard systems of implication
      i. strict implication
      ii. intuitionistic propositional logic
      iii. entailment and relevant implication
      iv. connexive implication
   d. nonstandard systems of quantification (see B2cv, E2a)

B. *Metalogic*

1. Logical Syntax

2. Logical Semantics
   a. basic semantics (denotation, extension/intension, truth, satisfiability, validity, completeness)
   b. theory of models
   c. special topics
      i. theory of definition
      ii. theory of terms (abstraction)
      iii. theory of descriptions
      iv. theory of identity
      v. logic of existence (existents and nonexistents) (see A3d, E2a)
      vi. logic of information and information-processing (see E3d)
3. Logical Pragmatics
   a. logical linguistics and the logical theory of natural languages (see B4)
   b. rhetorical analysis (Aristotelian 'topics'; the 'New Rhetoric' of Chaim Perelman)
   c. 'contextual implication' (in the sense of Grice)
   d. theory of informal (or material) fallacies
   e. unorthodox applications of logic
4. Logical Linguistics (see B3a)
   a. theory of structure (morphology)
   b. theory of meaning
   c. theory of validity

C. *Mathematical Developments*

1. Arithmetical
   a. algorithms
   b. theory of computability
   c. computer programming
2. Algebraic
   a. Boolean algebra
   b. lattice-theoretic logic
3. Function-Theoretical
   a. recursive functions
   b. Lambda conversion
   c. combinatory logic
4. Proof Theory (theory of axiomatizability, Gentzenization)

5. Probabilistic Logic (see E4b)
[6. Theory of Sets]
[7. Foundations of Mathematics]

D. *Scientific Developments*

1. Physical Applications
   a. quantum-theoretic logic
   b. theory of 'physical' or 'causal' modalities (see A3aii)
2. Biological Applications
   a. Woodger-style developments
   b. cybernetic logic
3. Social-Science Applications
   a. logic of norms (see E1b)
   b. logic of valuation
   c. legal applications

E. *Philosophical Developments*

1. Ethical Applications
   a. logic of action
   b. deontic logic (see D3a)
   c. logic of commands (logic of imperatives)
   d. logic of preference and choice (utility, cost, logical issues in the theory of games and decisions)
2. Metaphysical Applications
   a. logic of existence (see B2cv, A3d)
   b. chronological logic (tense-logic, change-logic, logic of process)
   c. logic of part/whole (mereology, the calculus of individuals)
   d. Leśniewski's 'ontology'
   e. constructivistic logic (logical reductionism, *Aufbau-ism*)
   f. ontology (in the sense of the nominalism-realism debate)
3. Epistemological Applications
   a. logic of questions (and answers)
   b. epistemic logic (belief, assertion, knowledge, relevance, 'about', and other intentional concepts)
   c. logic of supposition (hypothetical reasoning, counterfactuals)
   d. logic of information and information-processing (see B2cvi)
   e. inductive logic (see B4)

4. Inductive Logic (see E3e)
   a. logic of evidence and confirmation, acceptance (rules of acceptance)
   b. probabilistic logic (see C5)

APPENDIX B

A CONCISE BIBLIOGRAPHY OF PHILOSOPHICAL LOGIC

The coverage of this bibliography is restricted to the three regions of the map of Appendix A which we have designated as comprising the area of philosophical logic (viz., A3, B, and E). Even over this limited range, the listing is highly selective, indicating only one or two key systematic expositions in each case. It is worthy of note that the majority of entries (41 of 68) represent publications of the 1960s. The reader interested in further bibliographic data should consult these works in turn, and can find a comprehensive and up-to-date account of current work in all branches of logic in the Reviews sections of *The Journal of Symbolic Logic*.

(A3a) *Modal Logic* (Cf. Ch. III–IV)

G. H. von Wright, *An Essay in Modal Logic* (Amsterdam, 1951).

A. N. Prior, *Formal Logic* (Oxford, 1955).

S. A. Kripke, 'A Completeness Theorem in Modal Logic', *The Journal of Symbolic Logic* **24** (1959) 1–14.

S. A. Kripke, 'Semantical Considerations on Modal Logic', *Acta Philosophica Fennica*, fasc. 16 (1963) 83–94. (Cf. also J. Hintikka, *ibid.*, 65–81).

R. Feys, *Modal Logics* (ed. J. Dopp), (Louvain and Paris, 1965).

(A3b) *Many-Valued Logic* (Cf. Ch. VI)

J. B. Rosser and A. R. Turquette, *Many-Valued Logics* (Amsterdam, 1952).

A. N. Prior, *Formal Logic* (Oxford, 1955).

R. Ackermann, *An Introduction to Many-Valued Logics* (London, 1967).

A. A. Zinov'ev, *Philosophical Problems of Many-Valued Logic* (Dordrecht, 1963).

See also pp. 116–125.

(A3c) *Nonstandard Systems of Implication*

Wilhelm Ackermann, 'Begründung einer strengen Implikation', *The Journal of Symbolic Logic* **21** (1956) 113–128.

A. R. Anderson, 'Completeness Theorems for the System E of Entailment and the System EQ of Entailment with Quantification', *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* **6** (1960) 201–216.

A. R. Anderson and N. D. Belnap, Jr., 'The Pure Calculus of Entailment', *The Journal of Symbolic Logic* **27** (1962) 19–52.

(A3d) *Nonstandard Systems of Quantification* (Cf. Ch. VII–X)

P. T. Geach, *Reference and Generality* (Ithaca, 1962).

(B1) *Logical Syntax*

R. Carnap, *Logical Syntax of Language* (London, 1937).

(B2) *Logical Semantics* (Cf. Ch. II)

R. Carnap, *Introduction to Semantics*, (Cambridge, Mass., 1946).

R. Carnap, *Meaning and Necessity* (Chicago, 1958).

W. V. Quine, *Word and Object* (New York, 1959).

A. Tarski, *Logic, Semantics, Metamathematics* (Oxford, 1960).

(B3) *Logical Pragmatics*

H. P. Grice, 'Meaning', *The Philosophical Review* **66** (1957) 377–388.

S. Toulmin, *The Uses of Argument* (Cambridge, 1958).

R. M. Martin, *Towards a Systematic Pragmatics* (Amsterdam, 1959).

C. Perelman, *Traité de l'argumentation* (Paris, 1958).

C. Perelman, *The Idea of Justice and the Problem of Argument*, tr. J. Petrie (New York, 1963).

(B4) *Logical Linguistics*

J. Lyons, *Structural Semantics* (London, 1963).

N. Chomsky, *Aspects of the Theory of Syntax* (Cambridge, Mass., 1965).

J. J. Katz, *The Philosophy of Language* (New York, 1966).

(Ela) *Logic of Action*

G. H. von Wright, *Norm and Action* (London, 1963).

N. Rescher (ed.), *The Logic of Decision and Action* (Pittsburgh, 1967).

(Elb) *Deontic Logic* (Cf. Ch. XVI)

G. H. von Wright, *An Essay in Modal Logic* (Amsterdam, 1951).

A. R. Anderson, *The Formal Analysis of Normative Systems* (New Haven 1956); reprinted in N. Rescher (ed.) *The Logic of Decision and Action* (Pittsburgh, 1967).

(E1c) *Logic of Commands*

L. Bergstrom, *Imperatives and Ethics* (Stockholm, 1962).

N. Rescher, *The Logic of Commands* (London, 1966).

(E1d) *Logic of Preference and Choice* (Cf. Ch. XV)

S. Halldén, *On the Logic of 'Better'* (Uppsala, 1957; Library of Theoria, no. 2).

G.H. von Wright, *The Logic of Preference* (Edinburgh, 1964).

R.C. Jeffrey, *The Logic of Decision* (New York, 1965).

See also pp. 318–320.

(E2a) *Logic of Existence* (Cf. Ch. IX)

Dana Scott, 'Existence and Description in Formal Logic' in R. Schoenman (ed.), *Philosopher of the Century*, Essays in Honor of Bertrand Russell (London, 1967).

B. van Fraassen, 'The Completeness of Free Logic', *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* **12** (1966) 219–234.

(E2b) *Chronological Logic* (Cf. Ch. XII–XIII)

A.N. Prior, *Time and Modality* (Oxford, 1957).

G.H. von Wright, 'And Next', *Acta Philosophica Fennica*, fasc. 18 (1965) 293–304.

A.N. Prior, *Past, Present, and Future* (Oxford, 1967).

See also pp. 221–223.

(E2c) *Logic of Part/Whole*

A. Tarski, 'Appendix E' to J.H. Woodger, *Axiomatic Method in Biology* (London, 1937).

N. Goodman and H.S. Leonard, 'The Calculus of Individuals', *The Journal of Symbolic Logic* **5** (1940) 45–55.

(E2d) *Leśniewski's 'Ontology'*

E.C. Luschei, *The Logical Systems of Leśniewski* (Amsterdam, 1962).

(E2e) *Constructive Logic*

N. Goodman, *The Structure of Appearance* (Cambridge, Mass., 1951).

(E2f) *Ontology (Nominalism/Realism)*

W.V. Quine, 'On What There is', *The Review of Metaphysics* **2** (1948); reprinted in *idem, From a Logical Point of View* (Cambridge, Mass., 1953).

A. Church, 'The Need for Abstract Entities in Semantic Analysis', *Proceedings of the American Academy of Arts and Sciences* **80** (1951) 100–112.

N. Goodman, *The Structure of Appearance* (Cambridge, Mass., 1951).

(E3a) *Logic of Questions*

D. Harrah, *Communication: A Logical Model* (Cambridge, Mass., 1963).

N. D. Belnap, Jr., *An Analysis of Questions: Preliminary Report* (Santa Monica, 1963).

L. Aqvist, *A New Approach to the Logical Theory of Interrogatives*, Pt. 1 (Uppsala, 1965).

(E3b) *Epistemic Logic* (Cf. Chs. V, XIV)

J. Łoś. See the review of his Polish paper by H. Hiż in *The Journal of Symbolic Logic* **16** (1951) 58–59.

J. Hintikka, *Knowledge and Belief* (Cornell, 1962).

See also pp. 280–281.

(E3c) *The Logic of Supposition*

S. Jaśkowski, 'On the Rules of Suppositions in Formal Logic', *Studia Logica*, no. 1 (Warsaw, 1934).

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CHAPTER II

SELF-REFERENTIAL STATEMENTS

The standard resolution of the semantical paradoxes arising from self-referential statements is to dismiss these statements *en bloc* as meaningless. In a recent article, A. N. Prior has deplored this wholesale solution as too drastic, and urges a more selective procedure.[1]

Prior's approach – if I understand him aright – is to dismiss as *prima facie* meaningless only those self-referential statements which cannot consistently be classified as either true or false. This includes not only the various well known semantical paradoxes such as that of the Liar, but also the following interesting case (due, in its essentials, to John Buridan of Buridan's Ass fame): Messrs. A, B, C, and D make statements on a certain occasion, A and B both uttering some palpable truth (say: $1+1=2$), C a palpable falsehood (say: $1+1=1$), and D saying that just as many speakers speak truly as falsely on this occasion. (Thus if D's statement is *classed* as true, he speaks a falsehood; and if it is *classed* as false, he speaks a truth.) In such cases, Prior would reject the pivotal statements as meaningless specifically because they cannot viably be classed as true or as false – and *not* generically because they involve self-reference. Had Mr. 'Liar' said that his (self-same) statement was *true*, then – since no impossibility inheres in classing *this* statement of his as true – Prior would (I take it) be prepared to accept the self-referential statement as meaningful. Or again, had Buridan's last speaker said that *fewer* truths than falsehoods were spoken on the occasion in question, his self-referential could be classed as false without giving rise to difficulties, and would thus be meaningful on Prior's criterion.[2] Prior's solution thus has the important merit of liberality – it exiles self-referential statements from the domain of the meaningful not as a matter of inflexible policy, but only in cases of actual necessity.

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[1] 'On a Family of Paradoxes', *Notre Dame Journal of Formal Logic* **2** (1961) 16–32.

[2] An interesting if not strictly relevant case arises if D says that *at least three* truths are spoken. For this statement – which could feasibly be classed as false – is self-validating: if taken as true it *is* true.

One immediate – and of itself by no means unacceptable – consequence of this criterion is that not merely will certain self-referential statements be *meaningful*, but some of them will even have to be regarded as *necessary*. For example the statement 'There are false statements', symbolically '$(\exists p) \sim p$', will have to be regarded as a necessary truth. (It cannot be classed as false, since it can be inferred from its own denial; on the other hand, no difficulty ensues if it is accepted as true.)

So far so good. But now, as Prior points out, a further much more subtle complication must be introduced, namely that self-referential statements can be such that if certain preconditions fail to be satisfied these statements 'cannot even be made' (in Prior's language) or rather, they are only *conditionally meaningful* (as I would prefer to put it).

Suppose that Epimenides the Cretan says that nothing said by a Cretan is the case. Then we could readily class Epimenides' statement as false, though it could not possibly be true. But this, as Church has pointed out, commits us to accepting as true the *contradictory* of this statement, viz., that something said by a Cretan is true. Now since the only Cretan statement we have been told about is false, this true Cretan statement – whose existence we are thus committed to suppose –, must be some other statement. Thus, if we are to regard Epimenides statement as meaningful (and then false), we are committed to presuppose the existence of at least one true Cretan utterance (a contingent fact). Epimenides' statement is thus – on this approach – only conditionally meaningful. It is indeed conditionally L-false: (i.e., *logically* false – false on logical grounds alone). It will have to be classed as false whenever its meaningfulness-condition is assumed to be satisfied.

Similarly – and somewhat more unpleasantly – it is easy to devise an example of a conditionally L-true statement. Suppose that Mr. X makes (in Noplacese) the statement that someone has (at some time or other) made a false statement (in Noplacese). We cannot possibly class this statement of X's as false, for in doing so we *eo ipso* render it true. Thus if we are to regard the statement as meaningful we must class it as true. But it then entails the existence of a Noplacese utterance distinct from itself (viz., one that is false). Therefore, if we are to regard Mr. X's statement as meaningful (and thus true) we are committed to presuppose the existence of at least one false Noplacese utterance (a contingent fact). Mr. X's statement is thus only conditionally meaningful, but is also

conditionally L-true. It will have to be classed as true whenever its meaningfulness condition is assumed to be satisfied.

The disadvantage of Prior's approach is illustrated by these examples. In certain cases it leads to the consequence that there are statements whose very *meaningfulness* (and not merely truth or falsity) can hinge upon a matter of contingent fact. And moreover this contingent meaningfulness gives rise to the anomaly that there are *conditionally* L-true (and L-false) statements – statements which in the very logic of things must be true if meaningful at all, but whose meaningfulness requires some *purely contingent* precondition to be satisfied.

I confess to being much in sympathy with the spirit of Prior's approach of avoiding the somewhat Procrustean policy of dismissing self-referential statements *en bloc* as meaningless. Very possibly the advantages of such greater liberality could outweigh its having certain somewhat distasteful consequences. The existence of such consequences must, however, be recognized.

But there is an alternative policy towards self-referential statements which, in the spirit of Prior's approach, does not gainsay their status as *statements* (by abandoning them as meaningless), but rather salvages them by gainsaying their status as *self-referential.* On this approach, we would construe X's statements 'Any statement I make is false', and 'Some statement I make is true' as 'Any statement I make *apart from this one* is false' and 'Some statement I make *apart from this one* is true' respectively. Similarly, if X says 'Every statement Y makes is false' and Y says 'Every statement X makes is true' then we would construe these as:

X: Every statement Y makes *apart from those about this statement of mine* is false.

Y: Every statement X makes *apart from those about this statement of mine* is true.

Here once we view Y's statement as included within the exception clause of X's statement, and X's statement as included within the exception clause of Y's (or both), paradox is avoided. The policy is to gloss such potentially self-referential statements exceptively, rather than as literally universal.

On this approach, self-reference (direct or oblique) is disallowed, or rather, it is treated as an optical illusion. Seemingly self-referential statements are viewed as careless shorthand formulations of non-self-referential

ones, and are systematically construed so as to exclude outright self-reference. This policy has substantial merits. Retaining the (*prima facie*) self-referential statements as semantically meaningful, it avoids at one stroke not only the traditional paradoxes, but also the anomalies inherent in Prior's approach. But, admittedly, it purchases these advantages at the price of turning its back upon that root source of difficulty: the feature of outright or oblique *self-reference*.[3]

[3] This chapter is an expanded version of 'A Note on Self-Referential Statements' published in the *Notre Dame Journal of Formal Logic* **5** (1964) 218–220. For a different approach to the elimination of the paradoxes of self-reference see pp. 277–280 below.

CHAPTER III

MODAL RENDERINGS OF INTUITIONISTIC PROPOSITIONAL LOGIC

The two systems of non-standard propositional logic that have been most extensively studied to date are C.I. Lewis' systems of 'strict implication' and the intuitionistic propositional logic as systematized by A. Heyting. The relationship between these systems, which has now been explored for over a generation, is of substantial interest. The aim of the present chapter is both to summarize and to extend what is known about this relationship. Its linkages with the established systems of modal logic represent one of the most significant bridges between modern intuitionistic logic and other branches of the subject whose historical rootings go far deeper.[1]

The intuitionistic propositional calculus (IPC) of Heyting[2] rests upon the following eleven axioms:

(A1) $p \to (p \wedge p)$

(A2) $(p \wedge q) \to (q \wedge p)$

(A3) $(p \to q) \to [(p \wedge r) \to (q \wedge r)]$

(A4) $[(p \to q) \wedge (p \to r)] \to [p \to r]$

(A5) $q \to (p \to q)$

(A6) $[p \wedge (p \to q)] \to q$

(A7) $p \to (p \veebar q)$

(A8) $(p \veebar q) \to (q \veebar p)$

(A9) $[(p \to r) \wedge (q \to r)] \to [(p \veebar q) \to r]$

(A10) $\neg p \to (p \to q)$

(A11) $[(p \to q) \wedge (p \to \neg q)] \to \neg p.$

Here the symbols '$\to$', '$\wedge$', '$\veebar$', and '$\neg$' are used for intuitionistic implication, conjunction, disjunction, and negation, respectively.

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[1] This chapter is a revised version of a paper 'On Modal Renderings of Intuitionistic Propositional Logic' published in the *Notre Dame Journal of Formal Logic* **7** (1966) 277–280.

[2] Based on axioms first presented in HEYTING [1], but now more readily accessible in HEYTING [5], see p. 101.

Moreover, there are certain theses which Heyting in his book of 1956 specifically rejects as intuitionistically unacceptable:

(U1)	$(p \vee q) \rightarrow (p \veebar q)$[3]	
(U2)	$\sim p \rightarrow \neg p$	
(U3)	$p \veebar \neg p$	[See p. 99.]
(U4)	$\neg \neg p \rightarrow p$	[See p. 99.]
(U5)	$(p \rightarrow q) \veebar (q \rightarrow p)$	[See p. 99.]
(U6)	$\neg (p \wedge q) \rightarrow (\neg p \veebar \neg q)$	[See p. 100.]
(U7)	$(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow q)$	[See p. 101.]
(U8)	$\neg \neg (p \veebar q) \rightarrow (\neg \neg p \veebar \neg \neg q)$	[See p. 101.]

The symbols '$\sim$', '&', and '$\vee$' will be used for 'ordinary' (non-intuitionistic) negation, conjunction, and disjunction, respectively; and '$\supset$' will be used below for material implication.

Various 'dictionaries' for 'translating' statement schemata of IPC into the vocabulary of Lewis' systems of strict implication have been or can be discussed. We shall consider the five following possibilities ('$\Box$' represents the modality of necessity, '$-\!\!<$' the relation of *strict implication*).[4]

IPC-vocabulary	Lewis-vocabulary				
	D1	D2	D3	D4	D5
a (variable)	a	a	$\Box a$	a	a
$\neg S$ (S or T are arbitrary schemata)	$\sim \Box S$	$\Box \sim S$	$\Box \sim S$	$\sim \Box S$	$\Box \sim \Box S$
$S \rightarrow T$	$\Box S -\!\!< \Box T$	$S -\!\!< T$	$S -\!\!< T$	$\Box S \supset \Box T$	$\Box S \supset \Box T$
$S \veebar T$	$\Box S \vee \Box T$	$\Box S \vee \Box T$	$S \vee T$	$\Box S \vee \Box T$	$\Box S \vee \Box T$
$S \wedge T$	$S \& T$	$\Box S \& \Box T$	$S \& T$	$S \& T$	$\Box S \& \Box T$

Dictionary D1 was discussed by Kurt Gödel in his important paper of

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[3] Actually, (U1) and (U2) are not explicitly rejected by Heyting. Non-intuitionistic disjunction and negation are not mentioned in his book, and from the intuitionistic point of view no meaning can be given to them. The intuitionistic unacceptability of these two principles is clear from the informal discussion, however. For the former see p. 97 and for the latter pp. 18–19 and 97–98.

[4] It is assumed throughout this paper that the only translational changes are those specified by the 'dictionaries' under consideration.

1931, and dictionaries D3–D5 were discussed by McKinsey and Tarski in a paper of 1948 that extended Gödel's findings. Dictionary D2 is, so far as I know, novel, but is patently closely akin to D1 on the one hand, and D3 on the other.

The result established by Gödel is this, that if we take Lewis' system S4 as governing the modal concepts, then dictionary D1 translates all IPC-theorems into S4-theorems and all IPC non-theorems into non-theorems of S4. This result was extended with respect to dictionaries D3-D5 by McKinsey and Tarski.

These results, however, envisage only the translation of formulas expressed in purely intuitionistic symbolism. They do not apply when one introduces 'mixed' formulas in which both intuitionistic and classical connectives occur, such as the intuitionistically rejected (U1) and (U2). (It is supposed that in 'translating' such mixed formulas all the *non-intuitionistic* connectives are unaffected and remain unchanged.[5]) Indeed, D1 translates (U2) into a theorem of S4, and so do D4 and D5. D3 on the other hand translates (U1) into a theorem of S4.

The advantage of D2 *vis-à-vis* the other 'dictionaries' is that it translates *all* of the U-theses into non-theorems of S4. A seeming disadvantage is that it translates IPC axioms (A5) and (A7) into non-theorems of S4. But this disadvantage is not so serious as it at first seems. For if these two axioms were replaced by

(A′5) $\Box q \rightarrow (p \rightarrow q)$
(A′7) $\Box p \rightarrow (p \veebar q)$

or by some equivalent formulation using purely intuitionistic vocabulary, then they too would go over into theorems of S4. I am not certain that such a revision of (A5) and (A7) would be acceptable to intuitionists. But I can think only of considerations that would make it palatable. After all, the principal function of an intuitionistic thesis of the form '...→---' is, as Heyting's discussion makes clear, to inform us as to the inferences that

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[5] This procedure is adopted to assure that all of the standard, non-intuitionistic theses of the initial, mixed system go over into theorems. Certain trivial, pointless departures from this policy are possible by making use of standard equivalences (e.g., by such a rule as, 'Translate "$\sim S$" as "$\sim \sim \sim S$"') – but the introduction of any modal connectives must be eschewed.

can be made once '...' is *established* (i.e., is 'necessary' in some intuitionistically appropriate sense). And thus the added '$\Box$' in the antecedents of (A'5) and (A'7) seems entirely commensurate with intuitionistic objectives. It is also germane to recall the criticism of theses akin to the original (A5) as being 'paradoxes' of implication.[6]

It is worthwhile to raise the question of the consequences of strengthening the underlying modal system from S4 to S5. If, following Gödel's lead, we construe the $\Box$-modality as representing *provability* (being '*beweisbar*'), we should expect that something should 'go wrong' when this step is taken, in the sense that some of the unacceptable U-theses should now become theorems. (For the characteristic axiom of S5, viz., '$\sim\Box\sim\Box p \supset \Box p$' is clearly false under this construction – the fact that 'p' is not provably unprovable does not assure the provability of 'p'.) This expectation is met. For example, with all of the 'dictionaries', the transform of (U8) is a non-theorem of S4 that becomes a theorem in S5 (and if dictionary D2 is put aside, this is also the case with (U6) and (U7)).

The data on which our discussion has been based are summarized in the tabulation on p. 22:[7]

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[6] However, if the proposed change of axioms (A'5) and (A'7) were adopted, it would be necessary to augment the resulting set of axioms somewhat if the usual set of 'intuitionistically desirable' theses is to be forthcoming as theorems. For, as Heyting observes (p. 101 of reference [5]), it is desired to have

$$(\neg(p \wedge q) \wedge p) \rightarrow \neg q$$

as well as its logical consequence

$$(\neg\neg p \wedge \neg\neg q) \rightarrow \neg\neg(p \wedge q)$$

as intuitionistically acceptable theses. But it is readily shown that these are independent of the revised axiom-set. For if these theses are translated into modal vocabulary by our last-given 'dictionary' they – unlike the IPC axioms – go over into non-theorems of S5 – as is readily seen by the use of Lewis' 'Group III'.

[7] I wish to thank Miss Barbara Anne Hunt for her help in checking the data compiled in this tabulation.

THE STATUS OF THE MODAL 'TRANSLATIONS' OF INTUITIONISTIC THESES

	D1	D2	D3	D4	D5
Axioms[8]					
(A1)	T	T	T	T	T
(A2)	T	T	T	T	T
(A3)	T	T	T	T	T
(A4)	T	T	T	T	T
(A5)	T	N	T	T	T
(A'5)	T	T	T	T	T
(A6)	T	T	T	T	T
(A7)	T	N	T	T	T
(A'7)	T	T	T	T	T
(A8)	T	T	T	T	T
(A9)	T	T	T	T	T
(A10)	T	T	T	T	T
(A11)	T	T	T	T	T
Rejected Theses					
(U1)	N	N	T	N	N
(U2)	T	N	N*	T	T
(U3)	N*	N	N	N*	N*
(U4)	N*	N	N*	N*	N
(U5)	N	N	N	N	N
(U6)	N*	N	N*	N*	N*
(U7)	N*	N	N*	N*	N*
(U8)	N*	N*	N*	N*	N*

Key T = theorem of S4 (and S5)
N = non-theorem of S4 and S5
N* = non-theorem of S4 that becomes a theorem of S5

[8] The validity of the rules of inference (viz., *modus ponens* for implication and the inter-substitutability of mutual implicants) is of course also preserved by these translation-schemes.

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[2] C. I. LEWIS and C. H. LANGFORD, *Symbolic Logic* (New York, 1932; second edition, New York, 1959).

[3] Kurt GÖDEL, 'Eine Interpretation des intuitionistischen Aussagenkalküls', *Ergebnisse eines mathematischen Kolloquiums* **4** (1933) 39–40. Reporting results presented in 1931.

[4] J. J. C. MCKINSEY and Alfred TARSKI, 'Some Theorems About the Sentential Calculi of Lewis and Heyting', *The Journal of Symbolic Logic* **13** (1948) 1–15.

[5] A. HEYTING, *Intuitionism* (Amsterdam, 1956).

CHAPTER IV

A CONTRIBUTION TO MODAL LOGIC

1. THE CONCEPT OF MODALITY

A proposition is presented by a complete, self-contained statement which, taken as a whole, will be true or false: *The cat is on the mat*, for example. When such a proposition is itself made subject to some further qualification of such a kind that the entire resulting complex is itself once again a proposition, then this qualification is said to represent a *modality* to which the original proposition is subjected. The classical modalities, treated by logicians at least since the time of Aristotle (b. 384 B.C.)[1], revolve around the notion of truth itself:

It is *necessarily* true (or: *false*) that *p*.
It is *actually* true (or: *false*) that *p*.
It is *possibly* true (or: *false*) that *p*.

These three modalities – those of the *necessary*, of the *actual* (or *assertoric*), and of the *possible* – may be characterized as *alethic* modalities, that is, as relating to the notion of truth (Greek: *alēthē*).

There are many other kinds of modality in addition to the alethic modalities. Some important examples are:

Epistemic modalities, relating to knowledge (Greek: *epistēmē*) and belief
It is known (or: *X knows*) that *p*.
It is believed (or: *X believes*) that *p*.
It is accepted (or: *supposed, assumed*) that *p*.
It is anticipated (or: *expected*) that *p*.

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[1] Actually, in Aristotle himself one does not find the *term* modality (Greek: *tropos* = mode) at all, although the *idea* is explicitly present in his writings. In many of the Aristotelian commentators (especially Ammonius and Philoponus), a 'mode' can be presented by using any adverb to qualify the verb of a simple statement – e.g., by changing 'Socrates discourses' to 'Socrates discourses *well*'. See O. Hamelin, *Le système d'Aristote* (publié par L. Robin, Paris, 1920), p. 190.

Temporal modalities, relating to time

It is *sometimes* the case that *p*.
It is *mostly* the case that *p*.
It is *always* the case that *p*.
It *has always been* the case that *p*.
It *was yesterday* the case that *p*.

Boulomaic modalities, relating to desire (Greek: *boulomai*)

It is hoped (or: *X hopes*) that *p*.
It is feared (or: *X fears*) that *p*.
It is regretted (or: *X regrets*) that *p*.
It is desired (or: *X desires*) that *p*.

Deontic modalities, relating to duties (Greek *deō* = to require)

It ought to be brought about that *p*.
It ought to be avoided (or: *prevented*) that *p*.
It is forbidden to bring it about that *p*.
It is permissible to bring it about that *p*.

Evaluative modalities

It is a good thing that *p*.
It is a perfectly wonderful thing that *p*.
It is a bad thing that *p*.

Causal modalities

The existing state of affairs will bring it about that *p*.
The existing state of affairs will prevent (or merely: *will impede*) its coming about that *p*.

We must note also the existence of such binary (two-place) modalities as '*p* is more firmly believed than *q*', '*p* is more often the case than *q*', 'It is better that *p* than that *q*'. These illustrations should suffice to indicate the wide range over which the concept of modality finds application. The present chapter will, however, focus upon the alethic and epistemic modalities.

The theory of the alethic modalities of propositional *necessity*, *possibility* and their cognates is well known, and the logic of these concepts has been investigated extensively.[2] In an effort to devise a more flexible logical instrument, better suited for the application of modal logic in

[2] A fine introduction to modal logic is G. H. von Wright's book, *An Essay in Modal Logic* (Amsterdam, 1951).

certain philosophical contexts, G. H. von Wright has supplemented the familiar logic of absolute modalities by inaugurating a theory of conditional modalities.[3] Here, instead of the (absolute) modalities '$\Diamond p$' for 'p is *possible*' and '$\Box p$' for 'p is *necessary*', we find the concepts of *conditional* possibility and necessity, with '$P(p/q)$' to be interpreted as 'p is possible given q', and '$N(p/q)$' to be interpreted as 'p is necessary given q'.[4] It is the object of the present essay to introduce yet two other, new modalities. The first of these represents the idea of *conditional realization*, with '$A(p/q)$' standing for 'p is actual given q', and the second innovation is a *probability modality*, with '$L(p/q)$' standing for 'p is probable (or likely) given q'. It is our primary aim to present a detailed consideration of the meaning of these modalities, and of their formal logic. Also some brief observations on the uses and implications of these two innovations in modal logic will be offered.

2. CONDITIONAL REALIZATION

The idea of conditional realization arises out of a gap in von Wright's theory of conditional modalities. The traditional trichotomy of the propositional modalities of *necessity*, *possibility*, and *actuality* – reflected in the notations '$\Box p$', '$\Diamond p$', and '$\mathbf{T}p$' in modal systems – is not carried over into von Wright's system of conditional modalities. Instead, we find only the two modalities of conditional necessity, '$N(p/q)$' (p is necessary given q), and of conditional possibility, '$P(p/q)$' (p is possible given q). What is missing here, and lacking in a very obvious way indeed, is a third modality '$A(p/q)$' standing for 'p is *actual* (i.e., is the case) given q'. This modality must clearly be governed by the rules that: (1) $N(p/q)$ entails $A(p/q)$, (2) $A(p/q)$ entails $P(p/q)$, and (3) $A(p/q)$ and q, given conjointly, entail p.[5] The relationship that holds between 'p' and 'q' when this actuality-modality obtains between them, i.e., when '$A(p/q)$' is true, I designate as *conditional*

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[3] Von Wright's system was first presented in his paper 'A New System of Modal Logic', *Proceedings of the XIth International Congress of Philosophy* (Brussels, 1953), V, 59–63. A much expanded version of this paper is presented under the same title in von Wright, *Logical Studies* (London, 1957), pp. 89–126.

[4] Von Wright employs 'M' (for *möglich*) in place of 'P' (for *possible*).

[5] Here, and throughout, quotation-marks are omitted, and symbols used autonymously, where confusion cannot result.

realization. The justification for this designation of 'A' as an actuality-modality derives from the following considerations. Let '*t*' and '*f*' be propositional constants representing the truth-values *true* and *false*, respectively. Now we have: (1) $\Box p \leftrightarrow N(p/t)$, (2) $\Diamond p \leftrightarrow P(p/t)$, (3) $\Box \sim p \leftrightarrow N(f/p)$, and (4) $\Diamond \sim p \leftrightarrow P(f/p)$. These relationships provide the basis for a partial analogy between 'A' on the one hand, and 'N' and 'P' on the other, an analogy which provides the grounds for characterizing 'A' as an actuality modality. For it satisfies the conditions that $A(p/t) \rightarrow p$ and $A(f/p) \rightarrow \sim p$, and conversely.

These considerations suffice to indicate the character of conditional realization as a modal relationship 'intermediate' in logical strength between the modalities of conditional necessity and conditional possibility. Let us turn to a more detailed examination of the interpretation of this concept.

The intended meaning of the concept of conditional realization is suggested, though in a manner that is admittedly vague, by the phraseology: 'Whenever (the state of affairs characterized by) the proposition *q* is realized, (the state of affairs characterized by) the proposition *p* is realized.' This description is ambiguous, because it is rather indefinite regarding the specific nature of the *connection* between *q* and *p*. The following four possibilities appear to be left open by the phraseology here employed:

(1) The relationship between *q* and *p* is that of conditional necessity, so that we actually have N (*p*/*q*). (This includes the case of entailment, i.e., of *p* being a logical consequence of *q*.)

(2) The relationship between *q* and *p* consists in what I shall term *de jure* causality, in that there is a body of natural laws L (i.e., a group of inductively well-established laws of the natural sciences) such that *p* can be derived as a logical consequence from the body of information consisting of *q* and L conjointly.

(3) The connection of *q* to *p* consists in what I shall term *de facto* causality, in that it is an inductively-well-founded proposition, directly and simply, that a state of affairs of the type characterized by *p* is realized whenever circumstances of the type described by *q* occur. (Unlike (2) above, this includes those inductively well-founded empirical correlations that neither amount to nor are based on natural laws.)

(4) The connection between *q* and *p* is merely an 'accidental' one, in that it 'just so happens' that *q* and *p* are appropriately connected, without dependence upon any inductively well-founded generalization. For example, let '*q*' state the

circumstance that 'X is B's neighbor', where B is known to reside in London, and let 'p' describe the circumstance that 'X resides in London'. Now whenever the circumstance q is realized by a person, the corresponding event p is indeed also realized, but this is due solely to the purely contingent or accidental fact that B happens (presently) to reside in London, and does not involve recourse to any inductively well-founded fact.

It will be remarked that these four alternative connections between q and p are listed in order of decreasing logical strength of the connecting link obtaining between them. Now the concept of *conditional* realization, as here intended, is to be understood to include all three of the rule-governed cases (1)–(3), but will exclude the weakest case of a purely accidental connection, i.e., case (4). The relationship of conditional realization will thus obtain between q and p whenever p's realization given q derives either from purely logical, i.e., *deductive* considerations, or else from causal, i.e., *inductive* grounds. The only *prima facie* admissible case that is to be excluded is that of a merely 'accidental', rather than rule-governed connection: the applicability of the concept of conditional realization is intended and stipulated to be confined to these cases in which some specification of the idea of rule-governed connections is forthcoming. (I shall revert below to a discussion of the reason for this exclusion.)

In order to provide an exact and rigorous foundation for the concept of conditional realization, let us undertake to examine the rules which govern the formal logic of this concept. As the basis for a formal articulation of the logical theory here in question, the following six rules will be laid down to serve in the capacity of axioms for conditional realization:

(R1) If p is realized given q, and q is the case, then p:

$$[\mathrm{A}(p/q) \,\&\, q] \rightarrow p.$$

(R2) If p is necessary given q, then p is realized given q:

$$\mathrm{N}(p/q) \rightarrow \mathrm{A}(p/q).$$

(R3) If q is self-consistent, and not-p is realized given q, then p is not realized given q:

$$\mathrm{P}(q/q) \rightarrow [\mathrm{A}(\sim p/q) \rightarrow \sim \mathrm{A}(p/q)].$$

(R4) If r is necessary given p, then if p is realized given q, r will be realized given q:

$$N(r/p) \rightarrow [A(p/q) \rightarrow A(r/q)].$$

(R5) If p is realized given q, and r also, then p is realized given q and r (jointly):

$$[A(p/q) \,\&\, A(r/q)] \rightarrow A(p/q \,\&\, r).$$

(R6) If p is realized given q, and r is realized given p and q (jointly), then p and r are (jointly) realized given q:

$$[A(p/q) \,\&\, A(r/q \,\&\, p)] \rightarrow A(p \,\&\, r/q).$$

The *consistency* of this axiom system can be seen by interpreting 'A(.../---)' as '---$\supset$...', i.e., as '--- materially implies ...'.

In virtue of the intended meaning of 'A(.../---)', the informal, intuitive justification of each of these axioms is quite apparent. Of course, the intuitive acceptability of the axioms for the intended interpretation of 'A' does not entail that of the axiom-*system* as a whole, since this requires also the acceptability of their consequences, however remote. Such justification awaits elaboration of the consequences of these rules, and will in part be provided through the theorems to be exhibited.

Some of the leading consequences of R1–R6 will now be given. Such derived rules will be presented here without proof: the proofs are without exception elementary and straightforward. These theorems are listed in a natural deductive sequence. In each instance those previously established propositions that are needed for the proof are indicated:

(R7) $A(p/t) \rightarrow p$, by R1.
(R8) $A(f/p) \rightarrow \sim p$, by R1.
(R9) $A(t/q)$, by R2.
(R10) $A(p/f)$, by R2.
(R11) $A(p/p)$, by R2.
(R12) $A(p \,\&\, r/q) \rightarrow [A(p/q) \,\&\, A(r/q)]$, by R4.
(R13) $P(q/q) \rightarrow [A(p/q) \rightarrow P(p/q)]$, by R2, R3.
(R14) $[A(p/q) \,\&\, A(r/q)] \rightarrow A(p \,\&\, r/q)$, by R5, R6.
(R15) $A(p \,\&\, r/q) \leftrightarrow [A(p/q) \,\&\, A(r/q)]$, by R12, R14.
(R16) $[A(p/q) \vee A(r/q)] \rightarrow A(p \vee r/q)$, by R4.
(R17) $A(p \,\&\, r/q) \rightarrow A(p/q \,\&\, r)$, by R5, R12.
(R18) $[A(p/q) \,\&\, A(r/q \,\&\, p)] \rightarrow A(r/q)$, by R6, R12.

(R19) $\sim A(p/q) \rightarrow P(\sim p/q)$, by R2.
(R20) $P(q/q) \rightarrow [A(p/q) \rightarrow \sim A(\sim p/q)]$, by R3.
(R21) $[A(p/q) \,\&\, A(\sim p/q)] \rightarrow N(\sim q/q)$, by R3.
(R22) $P(q/q) \rightarrow [A(p/q) \rightarrow P(p/p)]$, by R2, R3.
(R23) $P(q/q) \rightarrow \{[A(p/q) \,\&\, A(r/q)] \rightarrow P(p/r)\}$, by R2, R3.
(R24) $P(q/q) \rightarrow \{A(p \vee r/q) \rightarrow [A(p/q) \vee A(r/q)]\}$, by R14, R20.
(R25) $P(q/q) \rightarrow \{A(p \vee r/q) \leftrightarrow [A(p/q) \vee A(r/q)]\}$, by R16, R24.

Here only those derived rules have been listed which are of the utmost importance either for an understanding of the logical nature of the relationship of conditional realization, or for the special purposes of the ensuing discussion. R3, R15, and R25 are especially revealing in indicating that conditional realization is in effect subject to most of the usual operations of conjunction, negation, and disjunction, respectively, as familiar from the ordinary logic of assertion. R18 is of special importance because it is only in this complex form that a transitivity relationship obtains for conditional realization, and *not* in the simple form:

$$[A(p/q) \,\&\, A(r/p)] \rightarrow A(r/q).$$

This feature, as we shall shortly see, is of special significance for the concept of conditional realization.

3. CONDITIONAL REALIZATION AND 'CAUSAL IMPLICATION'

In the fundamentals of its conception, the idea of conditional realization is akin to the idea of *causal implication* introduced by A. W. Burks.[6] Burks introduced the (unconditional) modalities 'is necessary on causal grounds' (which I will symbolize as 'Nc') and 'is possible on causal grounds' ('Pc'). Here we have: $Pc(p) \leftrightarrow \sim Nc(\sim p)$. And he then introduced a concept of 'causal implication' by the definition: $C(p/q) =_{Df} Nc(q \supset p)$. However, while there are indeed similarities between Burks' 'causal implication' and the present concept of 'conditional realization', there is one very fundamental point of difference between them which is brought out

[6] A. W. Burks, 'The Logic of Causal Propositions', *Mind* **60** (1951) 363–382 (see also the author's review of this in *The Journal of Symbolic Logic* **16** (1951) 277–278) and 'On the Presuppositions of Induction', *The Review of Metaphysics* **8** (1955) 574–611.

in the following passage of Burks' original paper:

> We intend... [by a statement that certain conditions causally imply the occurrence of some event] to assert that the conditions expressed... are *causally sufficient* to make... the statement describing the occurrence of the event true. By 'sufficient conditions' we mean a set of conditions complete with respect to negative properties as well as positive ones (i.e., counteracting causes must be explicitly mentioned) sufficient to cause the state of affairs expressed by the consequent.[7]

Thus Burks' concept of causal implication is intended to involve the requirement that the description of the causing circumstances in which an event is realized must be such as to *exclude explicitly all conceivable modification of these circumstances in which realization of the event is precluded.* In other words, when we have $C(p/q)$, the requirement must be met that there can be no statement r, logically compatible with q such that $C(\sim p/q\,\&\,r)$. This requirement of 'negative sufficiency' entails very serious and damaging consequences for the concept of causal implication.

For one thing, the requirement of negative sufficiency has the effect of rendering the concept of causal implication virtually impracticable of application. The description of the circumstances sufficient not only for realization of an event, but also complete with respect to the exclusion of all imaginable countervailing possibilities, is a task that is hopelessly cumbersome. It makes the concept of causal implication effectively impossible of application to actual cases. In science and in common life, while we do indeed often know that an event occurs under some describable set of circumstances, we are seldom if ever able to specify all of the conceivable further modifications of these circumstances that would interfere with the realization of the event in question. In the interest of having a concept that can be applicable in practice, it is therefore necessary to drop Burks' requirement of such 'negative sufficiency' in the characterization of causal circumstances.

I shall shortly advert to another and even more serious consequence of this requirement, namely that it in effect reduces this concept of an intendedly *causal* implication to *logical* implication, in that when we have $C(p/q)$, with strict fulfillment of the requirement of 'negative sufficiency', then q must be such as actually to *entail* p. For the moment, let it suffice to

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[7] A. W. Burks (*op. cit.*), pp. 368–382.

remark that while conditional realization is indeed in some respect similar to Burks' concept of causal implication, it differs from this in the details of the intended relationship between the propositions linked by it, and in particular, does *not* involve the requirement of 'negative sufficiency'.

It is quite illuminating to examine some of the propositions which *fail* to be rules of conditional realization and to consider the implications which these failures hold for the meaning of this concept. The most notable example of such a non-rule of conditional realization is:

(N1) $\quad (r \rightarrow q) \rightarrow [\mathrm{A}(p/q) \rightarrow \mathrm{A}(p/r)]$.

It may actually appear, on the first view, that this qualifies as a very plausible additional rule of conditional realization. Its acceptance would, however, prove fatal for the usefulness of the theory, for it would nullify its applicability to the intended purposes. Clearly, N1 has the consequence:

(N2) $\quad \mathrm{A}(p/q) \rightarrow \mathrm{A}(p/q \,\&\, r)$.

The implication of N2 for the theory of conditional realization is that it establishes the requirement that in stating the circumstances q in which an event p is realized, our description must be so complete that no conceivable (logically consistent) addition to the description of the circumstances q could possibly preclude the realization of p. In consequence, N2 leads to the requirement that any addition to q that can militate against realization of p must, if $\mathrm{A}(p/q)$ is given, be such as to be *logically incompatible* with q. Thus not only does N2 saddle us with the cumbersome requirement of 'negative sufficiency' in the statement of circumstances for conditional realization, its effect is actually to reduce the concept of conditional realization to that of logical entailment. Letting the 'r' of N2 be '$\sim p$', we see that N2 leads to: $\mathrm{A}(p/q) \rightarrow \mathrm{A}(p/q \,\&\, \sim p)$. But in view of R12, N2 also leads to $\mathrm{A}(\sim p/q \,\&\, \sim p)$. By R21, $\mathrm{A}(p/q) \rightarrow \mathrm{N}(\sim [q \,\&\, \sim p]/q \,\&\, \sim p)$. But $\mathrm{N}(\sim s/s)$ is possible only if s selfcontradictory, and if $\mathrm{A}(p/q)$ entails that $q \,\&\, \sim p$ is selfcontradictory, then it effectively entails $\mathrm{N}(p/q)$. Thus N2 has the effect of reducing conditional realization to conditional necessity, i.e., to entailment.

These considerations serve to elucidate still further the differences between conditional realization and A. W. Burks' concept of causal implication. In Burks' system, the counterparts of N1 and N2, with 'C' in place of 'A', obtain, laying that system open to the foregoing line of objection.

Again, analogous considerations require us to reject the analogue of Burks' simple transitivity rule, i.e., to reject:

(N3) $[A(p/q) \& A(r/p)] \rightarrow A(r/q)$.

In its place, we can have only the weaker transitivity relationships represented by R4 and R18.

N1–N3 serve to clarify the contrast between conditional realization on the one hand, and Burks' system of causal implication on the other, by spelling out in an explicit way the unfortunate implications of the requirement of 'negative sufficiency' already alluded to above.

In view of these differences, and in particular in view of the failure of N1 to be an acceptable assertion in the logic of conditional realization, it appears that this concept is not definable in terms of the simple, unconditional causal modalities that qualify statements of material implication, along the line of Burks' definition of causal implication. Rather, the concept of conditional realization is analogous to the purely logical *conditional* modalities, of the type studied by von Wright.

4. THE LIKELIHOOD MODALITY

The modal operator **T** represents an actuality modality: '**T***p*' means '*p* is definitely and actually true'. This is to be construed so that we have **T***p*→*p*, though not necessarily conversely. It is an interesting addition to the standard alethic modalities of truth, possibility, and necessity (**T***p*, $\Diamond p$, $\Box p$), to consider also the *likelihood modality*:

Op for 'probably *p*' or 'it is likely that *p*'.

We would, of course expect to have the chain of relationships:

$\Box$ **T***p* → *p*
T*p* → *Op*
Op → $\Diamond p$.

The inverse of each of these implications would, of course, fail to represent a general truth.

It would seem plausible that by and large a logical principle would hold for *O* if the corresponding principle holds for $\Diamond$. For example, it

might be noted that we have

$$\begin{array}{ll} \mathbf{T}p \rightarrow \Diamond p & \mathbf{T}p \rightarrow Op \\ \Diamond(p \,\&\, q) \rightarrow \Diamond p & O(p \,\&\, q) \rightarrow Op \\ \Diamond p \rightarrow \Diamond(p \vee q) & Op \rightarrow O(p \vee q) \end{array}$$

and the failure of

$$(\Diamond p \,\&\, \Diamond q) \rightarrow \Diamond(p \,\&\, q)$$

to obtain is matched by that of

$$(Op \,\&\, Oq) \rightarrow O(p \,\&\, q).$$

But this plausible correspondence fails of universality, since

$$\sim \Diamond p \rightarrow \Diamond \sim p$$

holds good, but the corresponding O-principle, namely

$$\sim Op \rightarrow O \sim p$$

fails, since there will be some propositions that are 'just as likely as not'.

This (unconditional) probability modality is not definable in terms of the other modalities, nor are they definable in terms of it.[8]

The understanding of the probability modality itself, as well as subsequent explanations and expositions regarding it, will be greatly facilitated by use of the concept of a probability measure for statements. A real-valued function $\mathrm{pr}(p)$ defined for all statements p belonging to a body of discourse D is called a *probability measure* on D if the following three conditions are satisfied:

(PR1) If p is in D, $\mathrm{pr}(p) \geqslant 0$.

(PR2) If $\sim p \vdash p$ (i.e., if p is necessary, $\Box p$), $\mathrm{pr}(p) = 1$.

(PR3) If $p \vdash \sim q$ (i.e., if p and q are incompatible, $\sim \Diamond [p \,\&\, q]$), then $\mathrm{pr}(p \vee q) = \mathrm{pr}(p) + \mathrm{pr}(q)$.

These three conditions are in fact axioms for the probability function, and assure that pr obeys all of the usual rules of the calculus of probability.[9]

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[8] This is readily shown by means of the probabilistic interpretation of modalities shortly to be discussed.

[9] See P. R. Halmos, 'The Foundations of Probability', *American Mathematical Monthly* **51** (1944) 493–510.

Specifically, when $\mathrm{pr}(q) \neq 0$, we can introduce a measure $\mathrm{pr}(p/q)$ of the conditional likelihood of p on q:

$$\text{(CP)} \qquad \mathrm{pr}(p/q) = \frac{\mathrm{pr}(p \,\&\, q)}{\mathrm{pr}(q)}.$$

This probability measure for statements is identical with the concept of a measure of *degree of confirmation* among statements. On this subject the reader is referred to R. Carnap's *Logical Foundations of Probability* (Univ. of Chicago Press, 1950). Several methods exist for the actual computation of numerical values of such a measure for statements within languages of various (generally quite simple) types, and Carnap's book should be consulted for details and for references to the literature. However, no details regarding specific pr measures will be required here. For our present purposes, the notion of a probability measure for statements will be used in only a very general way, in order to simplify problems of explanation and exposition.

In terms of such a measure of statement probabilities, we can now interpret 'Op' as amounting, in effect, to $\mathrm{pr}(p) > \frac{1}{2}$. Since $\Box p$ entails $\mathrm{pr}(p) = 1$ (though not necessarily conversely) and $\mathrm{pr}(p) > 0$ entails $\Diamond p$ (though not necessarily conversely), we have $\Box p \rightarrow Op$ and $Op \rightarrow \Diamond p$, as desired. We further have $Op \rightarrow \sim O \sim p$ (though not necessarily conversely, since $\sim O \sim p$ amounts to $\mathrm{pr}(p) \geqslant \frac{1}{2}$, with the weak inequality). Moreover, since we construe $\mathbf{T}p$ 'p is definitely true (actual)' as amounting in effect to $\mathrm{pr}(p) = 1$, it will obviously be the case that $\mathbf{T}p \rightarrow Op$.

Consequences of particular interest ensue upon returning the discussion to the sphere of conditional modalities. Here we introduce the conditional probability modality $\mathrm{L}(p/q)$, to be read, and interpreted, as 'p is probable (likely), given q'. In terms of our conception of statement probabilities this is to be taken to amount to the requirements that the conditional probability of p on q is greater than one-half, i.e., $\mathrm{pr}(p/q) > \frac{1}{2}$.

To begin with, it is of interest to observe that the absolute probability modality is readily defined in terms of its conditional cognate. On analogy with the relationships $\Box p \leftrightarrow \mathrm{N}(p/t)$ and $\Diamond p \leftrightarrow \mathrm{P}(p/t)$, we now have: $Op \leftrightarrow \mathrm{L}(p/t)$.

Furthermore, just as we have,

$$(1) \qquad [\mathrm{N}(p/q) \,\&\, \Box q] \rightarrow \Box p,$$

and (from section 2 above),

(R1) $[A(p/q) \& q] \rightarrow p$,

so, as a characteristic rule of inference governing the modality of conditional probability we have the rule:

(L1) $[L(p/q) \& \mathbf{T}q] \rightarrow Op$.

The rule L1 is comparable to the rule R1 for conditional realization: in weakening the antecedent from the relationship of conditional realization to that of probability, we correspondingly weaken the status of the consequent from that of factuality to that of likelihood.

The rationalization of this rule in terms of the intended meaning of the concepts involved is as follows: If $L(p/q)$, then $\mathrm{pr}(p/q) > \frac{1}{2}$. Therefore $\mathrm{pr}(p \& q) > \frac{1}{2}\mathrm{pr}(q)$. But if q is the case, then $\mathrm{pr}(q) = 1$. Thus $\mathrm{pr}(p) \geqslant \mathrm{pr}(p \& q) > \frac{1}{2}$. Q.E.D. We shall obviously also have

$$[N(p/q) \& Oq] \rightarrow Op$$

as another analogue of (1) and R1.

Some further examples of rules for the (conditional) probability are as follows:

(L2) $\sim \Diamond q \rightarrow \sim L(q/p)$.
(L3) $L(p/q) \rightarrow \sim L(\sim p/q)$.
(L4) $[L(p/q) \& \Box(p \rightarrow r)] \rightarrow L(r/q)$.
(L5) $[\Diamond p \& \Box(p \rightarrow q)] \rightarrow L(q/p)$.
(L6) $[\Diamond p \& L(q/p)] \rightarrow \Diamond(p \& q)$.

The acceptability of these rules can be checked in terms of the probabilistic interpretation of the probability modality. Needless to say, this listing is not exhaustive, and other rules for the probability modality are readily conceived in terms of the probabilistic interpretation, and are easily tested by its means.

5. CAN CONDITIONAL REALIZATION BE CONSTRUED PROBABILISTICALLY?

The character of our explication of the probability modality naturally raises the question if conditional realization cannot also be construed in

probabilistic terms. There immediately comes to mind the possibility of interpreting the relation of conditional realization $A(p/q)$ as amounting to the probabilistic relationship $\mathrm{pr}(p/q)=1$, so that p is considered as being realized conditionally relative to q if the conditional probability of p on q is unity (one). Indeed it was a desire not to preclude this possibility which provided one reason why purely 'accidental' relationships were (in the foregoing discussion) excluded from the intended meaning of conditional realization. (Such accidental connections cannot be viewed as providing a sufficient basis for assigning a value of unity to the conditional probability in question.)

However attractive such a probabilistic conception of conditional realization may appear on first view, it cannot be accepted in an unqualified way. For consider the proposed definition

(D1) $\quad A(p/q)$ FOR $\mathrm{pr}(p/q)=1$

according to which '$A(p/q)$' is to amount to: 'the conditional probability of p given q is 1'. This has the consequence that various rejected N-theses will obtain. As regards N2, for example, it is readily shown that

$$\mathrm{pr}(p/q)=1 \text{ or equivalently } \mathrm{pr}(\sim p \,\&\, q)=0$$

entails

$$\mathrm{pr}(p/q \,\&\, r)=1 \text{ or equivalently } \mathrm{pr}(\sim p \,\&\, q \,\&\, r)=0.$$

However, consider as an alternative to D1 the mixedly probabilistic and non-probabilistic definition:

(D2) $\quad A(p/q)$ FOR $(q \rightarrow p) \,\&\, (\mathrm{pr}(p/q) > 1-\varepsilon)$.

Here the probabilistic clause (viz., that $\mathrm{pr}(p/q)>1-\varepsilon$) in effect stipulates that p is *virtually certain* given q. The definition at once guarantees the essential features of the relationship of conditional realization: that $A(p/q)$ be stronger than either $q \rightarrow p$ and $L(p/q)$, that it be weaker than $N(p/q)$, and the like. Thus on this construction of the matter '$A(p/q)$' is to amount to: 'q implies p and moreover p is virtually certain given q'. This definition guarantees satisfaction of R1–R6 and yet does not lead to N1–N3. (Of course, acceptance of D2 in contrast to D1 clearly presupposes that we construe the implication relationship '$q \rightarrow p$' in a way that does not simply equate this with: $\mathrm{pr}(p/q)=1$.)

6. CONCLUSION

In recent logical research an impressive body of evidence has been built up that modal logic provides a powerful instrumentality for the investigation and clarification of philosophical problems. While there is little need for modalities in those branches of logic which have been cultivated primarily with a view towards mathematical applications (where modal ideas play no role), it appears, quite to the contrary, that modalities are virtually indispensable in philosophical logic. I wish, in concluding, to give two very brief indications of the potential usefulness of the foregoing considerations in modal logic for the investigation of philosophical questions.

My first point relates to a philosophical application of the concept of conditional realization. This concept can be used in an explication of counterfactual conditionals. In many or most counterfactual relationships between two statements, i.e., 'p would be so if q were so', the counterfactual condition q must be causally relevant to the realization of p in some genuine and appropriate sense. It would seem that conditional realization can provide the requisite evidential bond between the statements of a counterfactual conditional, making possible an analysis of counterfactuals in terms of conditioned realization, on close analogy with A. Burks' proposed analysis in terms of causal implication.[10] That is, we would construe 'if p had been the case, then q would have been the case' as $\Diamond p \,\&\, {\sim}p \,\&\, \mathrm{A}(q/p)$, i.e., as amounting, in effect, to a statement that: (1) the counterfactual hypothesis (i.e., p) is indeed *counterfactual* (i.e., is false), (2) an assumption of the denial of the counterfactual hypothesis is possible, and finally (3) there is an evidential relationship (viz., conditional realization) which assures us that, given the counterfactual hypothesis, its purported consequence would in fact ensue. In this way, the concept of conditional realization offers real promise as a factual concept capable of providing the basis of an analysis of contrary-to-fact conditionals.

Another philosophical application of the foregoing considerations in modal logic relates to the concept of evidence, which is the central idea of the theory of inductive reasoning, where our interest is in *good* but

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[10] See Burks' papers cited above. I have already tried to set forth in Section 3 the grounds of my view that Burks' analysis cannot be accepted as it stands.

not necessarily – as in deductive logic – *conclusive* grounds or reasons. One of the main senses of the concept of evidence is surely that a statement p is 'evidence' for a statement q if the assumption of p as hypothesis renders q *more likely than not*. This evidence concept, which I have elsewhere called *presumptive evidence*,[11] amounts to the condition that $\mathrm{pr}(q/p) > \mathrm{pr}(\sim q/p)$, which is readily shown to be equivalent with the condition $\mathrm{pr}(q/p) > \frac{1}{2}$. It therefore appears that the (conditional) probability $\mathrm{L}(q/p)$ is tantamount in meaning and intent to one of the principal interpretations of the idea of evidence. Thus the theory of the (conditional) probability modality has the potential of furnishing the epistemological basis, within an articulated system of modal logic, for the analysis of one of the central concepts of the theory of inductive reasoning, viz., the concept of evidence.

It would seem, therefore, that conditional realization and the probability modality are of importance not only as modal concepts of logical interest in their own right, but also as furnishing the conceptual basis for significant applications in philosophical analysis.[12]

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[11] 'A Theory of Evidence', *Philosophy of Science* **25** (1958) 83–94.

[12] This chapter is a revised version of a paper of the same title published in *The Review of Metaphysics* **12** (1958) 186–199.

CHAPTER V

EPISTEMIC MODALITY: THE PROBLEM OF A LOGICAL THEORY OF BELIEF STATEMENTS

1. INTRODUCTION

This discussion is of an exploratory nature. It does not attempt the presentation of any accomplished theory of the logic of belief statements. Rather, its objectives are of a more limited scope. The present chapter merely attempts to examine some of the problems and difficulties that confront the construction of such a theory. Despite this limitation, the purpose of these considerations is constructive rather than negative, because their main aim is to determine some of the characteristic features which must inevitably form a part of any adequate theory of belief able to overcome these difficulties.

The philosophical importance of such a theory is a matter on which it is doubtless superfluous to dwell at length. Suffice it here to quote a brief excerpt from Bertrand Russell's discussion of the belief-concept:

> The whole intellectual life consists of beliefs, and of the passage from belief to another by what is called 'reasoning'. Beliefs give knowledge and error; they are the vehicles of truth and falsehood. Psychology, theory of knowledge, and metaphysics revolve about belief, and on the view we take of belief our philosophical outlook largely depends.[1]

Thus more than theoretical value attaches to the logic of belief, in view of the central role which this concept is bound to play in a great many areas of philosophical interest.

2. THE EPISTEMIC MODALITY OF BELIEF

In the theory of modality, a case of special difficulty and interest is posed by the *epistemic* modalities. The specific modality upon which our discussion will focus is that of *belief*, that is, a binary modality involving both a believing person and a believed proposition. Correspondingly, we

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[1] *Analysis of Mind* (London, 1921), p. 231.

introduce the modality:

$B(x, p)$ for 'x believes that p'.

A word must be said about the matter of interpretation: how are we to understand 'x believes that p is the case'? Three major alternatives must be canvassed, viz.,:

(i) *x acts as though p* were the case

(ii) *x potentially overtly accepts p* in the sense that he would, if appropriately asked, indicate (if he is being candid) his subscription to this proposition

(iii) *x is committed to the acceptance of p by propositions to which he subscribes*, even though he may himself fail to recognize – and may even disavow – these implicit commitments.

The considerations at issue here are as follows: (1) The behavioristic conception (i) is not a very helpful basis for the conceptual analysis of beliefs – the man who 'acts as though' his house drains were made of copper does not act (under virtually all standard circumstances) differently from the man who believes them to be made of zinc, and indeed neither of them may in fact have given the matter many moments of consideration.[2] (2) The overt acceptance approach (ii) cannot take us very far: it has us depart the precincts of psychological fact too late to do much useful work in those of logical theory. The man who accepts p and $p \rightarrow q$ may be too foolish (or ignorant, or hasty, or 'inconsistent') to accept q. (3) Thus, in the interests of developing a workable theory it would seem that we are pretty well forced to the implicit-commitment approach (iii). In fact, however, we should like to occupy a middle ground between (ii) and (iii), in a way shortly to be explained.

3. NON-RULES FOR A THEORY OF BELIEF STATEMENTS

Perhaps the most interesting aspect of the logic of belief-statements, and

[2] A great many complexities inhere in the conception of 'acting as though'. The man who 'acts as though' he had a headache simply acts the way people generally do when they have headaches. But the man who 'acts as though' the building were on fire acts the way people generally do *when they believe* the building to be on fire. In many or most cases 'acting as though' involves an inherent reference to beliefs. This is why it seems to me unworkable to adopt alternative (i) and explicate the concept of belief in terms of 'acting as though'.

certainly the primary source of its complexity, derives from the departures from ordinary (assertory) sentential logic that are necessitated by the, as it were, 'irrational' or 'illogical' nature of belief. This is most clearly brought to view by considering some of the propositions that must *not* be included in a logical theory of belief statements. The aim of a theory of belief statements is to provide a systematic articulation of the rules that govern the formal logic of this relationship. To indicate some of the problems that arise in the construction of such a theory, it is useful first to examine some of the *non-rules* of the logic of belief, i.e., rules which must be *excluded* from an adequate theory. Consider, for example, a rule to the effect that if a person does not believe a certain statement, then he believes the negation of that statement:

(N1) $\quad \sim \mathrm{B}(x, p) \rightarrow \mathrm{B}(x, \sim p)$.

This has the unacceptable consequence that, for any statement p whatsoever, a person must either believe p, or else must believe its negation, $\sim p$.[3] Acceptance of N1 would thus rule out, without further ado, the possibility both of indifference or ignorance as between contradictories, and of a deliberate suspension of judgment.

These consequences of an acceptance of (N1) are thus not acceptable, and so this rule itself has to be rejected. Note, however, that the converse of this rule, namely

$$\mathrm{B}(x, \sim p) \rightarrow \sim \mathrm{B}(x, p)$$

is a plausible, and indeed inevitable principle for the rationalized construction of belief that undergirds our formal theory.

Another possible rule of a theory of belief whose rejection is also mandatory is a thesis to the effect that a person must believe the logical consequences of a believed statement:

(N2) $\quad [\mathrm{B}(x, p) \,\&\, (p \rightarrow q)] \rightarrow \mathrm{B}(x, q)$.

Acceptance of this rule would commit us, *inter alia*, to the patent absurdity that a person cannot fail to draw any of the appropriate logical infer-

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[3] Where confusion cannot result, I adopt the practice of autonymous use of symbols. Also, it should be noted that throughout, the arrow '→' is used to represent logical entailment or strict implication (and *not* material implication).

ences from a believed proposition. It can, surely, happen that of two (entirely rational and reasonable) people, each of whom believes the *axioms* of a theory (say the theory of numbers), one believes and the other disbelieves some complex and remote proposition that is purportedly a consequence of these axioms (e.g., Fermat's Last Theorem). Clearly it is only when a fact of entailment is explicitly *recognized* by a person, that the entailed consequence must also be accorded the status of belief. Thus a close analogue of (N2) would indeed be quite acceptable, namely the thesis:

$$[\mathrm{B}(x, p) \,\&\, \mathrm{B}(x, p \rightarrow q)] \rightarrow \mathrm{B}(x, q).$$

But it is clear of itself, and clearer yet from a comparison of (N2) with this just-stated thesis that (N2) is *not* acceptable as it stands. When the logical consequences of a believed proposition fail to be remarked or noted, their logical status *per se* wholly lacks the power to compel belief. For reasons such as this, it is apparent that (N2) must be excluded from the formal logic of belief.

Considerations of a wholly analogous kind show that it is necessary also to reject the thesis that a person must believe a statement logically equivalent with a believed statement:

(N3) $\quad [\mathrm{B}(x, p) \,\&\, (p \leftrightarrow q)] \rightarrow \mathrm{B}(x, q).$

For (N3) again founders on the failure to accord adequate recognition to the possibility that the logical relationship in question here (viz., the logical equivalence), can obtain and yet go altogether unremarked.

Similarly, we must on analogous grounds reject the claim that a person cannot believe a statement incompatible with a believed statement:

(N4) $\quad (\mathrm{B}(x, p) \,\&\, (p \rightarrow \sim q)] \rightarrow \sim \mathrm{B}(x, q).$

For, here also, when the fact of incompatibility goes unremarked this rule is invalidated since there arises a possibility of a conjoint belief in incompatibles.

The principle which has guided the foregoing rejections of N2–N4 is that such logical relations as entailment, equivalence, and incompatibility can obtain, but yet go unnoticed. When these logical relationships go unrecognized, they can plainly have no prescriptive force for determination of belief. In building up a formal theory of belief we may indeed assume that when logical relations are noted or recognized they must have

a formative influence upon reasoned belief. And we can (since we are concerned to build up a logical theory that is to apply to solely rational and reasonable persons) assume that certain of the most obvious logical relationships cannot go unrecognized. (Thus we might, e.g., accept (N3) with '$\sim\sim p$' in place of 'q'.) But of course we cannot in general postulate that *actual* logical relationships must necessarily be recognized. For there is no justification whatever for assuming that the individuals with whom our theory concerns itself are *omniscient* with respect to the domain of logic.

Herein, then, we have the root reason for the odd-seeming and complex character of the logical theory of belief. Many relationships which familiarity with the logic of assertion has accustomed us to conceive as holding among propositions, cannot justifiably be carried over into the logical theory of belief statements. Logical relationships which do in general hold among propositions *per se*, need not hold, *ipso facto*, among propositions *qua* believed.

4. THE CRITERION-PROBLEM FOR A THEORY OF BELIEF STATEMENTS

The foregoing examination of some of the possible rules for the belief-relationship which must be *excluded* from an acceptable logical theory of belief-statements, leads naturally to the question of what rules *do* in fact hold for this concept. A convenient starting-point for a discussion of this question is afforded by a proposal due to A. Pap,[4] who offers (in effect) a system based on six rules, taken as providing the postulates for an axiomatic theory of belief statements. Four of these rules may be stated as axioms:

(P1) $[\mathrm{B}(x, p) \,\&\, \mathrm{B}(x, p \rightarrow q)] \rightarrow \mathrm{B}(x, q)$

(P2) $\mathrm{B}(x, \sim p) \rightarrow \sim \mathrm{B}(x, p)$

(P3a) $\mathrm{B}(x, p \,\&\, q) \rightarrow \mathrm{B}(x, p)$

(P3b) $\mathrm{B}(x, p \,\&\, q) \rightarrow \mathrm{B}(x, q)$.

Pap's two remaining rules are:

(R1) That if we have it that, for *all* individuals x, $\mathrm{B}(x, p) \rightarrow \mathrm{B}(x, q)$ then this constitutes a sufficient condition for saying that p entails (or rather, strictly implies) q, that is, $p \rightarrow q$.

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[4] 'Belief and Propositions', *Philosophy of Science* **24** (1957) 123–136.

(R2) That no one is omniscient, i.e., that there is no individual x such that for this individual we have it that, for any and every proposition p, p is the case whenever $B(x, p)$.

A brief discussion of these rules is in order:

(1) The main group of axioms consists of those which – like (P1), (P3a), and (P3b) – have it that a person can make certain elementary inferences among believed propositions.

(2) One of the axioms, namely (P2), should be viewed in the light of a consistency principle. Asserting the incompatibility of $B(x, p)$ and $B(x, \sim p)$, it marks the theory at issue as one of *rational* belief, precluding the holding of incompatible beliefs.

(3) I cannot with any assurance guess at the precise intent of Pap's rule (R1). Probably, however, it is the same as with our rule (B3) below.

(4) Rule (R2) – I should like to call it *the antitheological postulate* – however plausible it may be, is clearly not a proper component of a *logical* theory of rational belief as such.

Another significant feature of Pap's set of axioms regards axioms which separate the logic of belief from the logic of assertion, by providing for a differentiation between belief statements and the purely factual statements corresponding to them. (R2), or equivalently, $(\forall x)(\exists p)\Diamond[B(x, p) \& \sim p]$, falls into this group. (Note that, in the absence of (R2), (P1)–(P3) admit of the interpretation that '$B(x, p)$' be taken as simply 'p', for on this interpretation each of these axioms is transformed into a correct assertion of modal logic.) Moreover, Pap's list includes axioms which assure that the possibility of belief-divergences is kept open as long as this is not precluded by purely logical considerations. (R1), or equivalently $\sim(p \rightarrow q) \rightarrow (\exists x)\Diamond[B(x, p) \& \sim B(x, q)]$, falls into this class.

However there is manifestly no way of avoiding the question of whether there are not, within each of the aforementioned groups, other, equally plausible theses for a theory of belief, rules neither equivalent with, nor derivable from those given. For example, why should we not strengthen R1 to:

(1) $\sim(p \rightarrow q) \rightarrow (\exists x)\Diamond[B(x, p) \& B(x, \sim q)]$?[5]

[5] Actually, (R1) is almost certainly too strong as it stands. Note that it is equivalent to:

(i) $\sim(p \rightarrow q) \rightarrow (\exists x)\Diamond[B(x, p) \& \sim B(x, q)]$.

If we assume, as surely we must, that, if p is a contingent proposition, we can have

Again, why should not (R2) be weakened to $(\exists x)(\exists p)\Diamond[B(x, p) \& \sim p]$? Most importantly of all, why should not Group (I) be expanded to include a great variety of other elementary inferences, such as $B(x,p) \rightarrow B(x, \sim\sim p)$, $B(x, p) \rightarrow B(x, p \vee q)$, $B(x, p \& q) \rightarrow B(x, q \& p)$, etc., etc. Indeed, what is to prevent expansion of this group to include belief-analogues for the whole of elementary propositional logic?

My object in raising these questions is not so much to advance some proposal for a specific alternative to Pap's axiomatic theory of belief. Rather, it is to bring home the necessity for developing, in advance of the enunciation of any such theory at all, a suitable *criterion of acceptability* or *criterion of adequacy* for the rules of a theory of belief statements, a criterion that can serve both as a test for the screening of proposed rules, and as a guide for the construction of the theory. In the absence of such a guiding criterion, we are merely groping in the dark in the task of theory-building. Only with the prior development of at least a partial criterion of acceptability does it become possible to make a coherent and systematic attempt to construct a set of rules for the theory of belief.

5. A PROPOSAL

In the interests of devising what seems to be a plausible theory of rational belief, let us lay down the following rules:

(B1) If p is a *self-inconsistent proposition*, that is, if $\Box \sim p$, then we must never have it that $B(x, p)$.

(B2) Whenever p is an 'obvious consequence' of two propositions q and r in the sense that p can be obtained as a conclusion from q and r as premisses by only n ($n = 1, 2, 3$ or some other small number) inferential steps, then from $B(x, q)$ and $B(x, r)$ as premisses we may infer $B(x, p)$ as the conclusion.[6]

$\sim(p \rightarrow [p \& q])$, then (i) entails, $(\exists x)\Diamond B(x,p)$. Thus (i) has the consequence of asserting that any contingent proposition is a possible object of belief for some person. This, it seems clear, can scarcely be advanced as a strictly *logical* truth. Thus (P1) must surely in any event be weakened to:

(ii) $\Box(\forall x)[B(x,p) \rightarrow B(x,q)] \rightarrow (p \rightarrow q)$.

[6] This conception of an 'obvious consequence' has a close relationship to the conception of a 'surface tautology' introduced into the logic of belief (for purposes quite similar to ours) by Jaakko Hintikka. See his papers '"Knowing Oneself"

(B3) It is never to happen that from $B(x, p)$ we can *deduce* $B(x, q)$ unless q is an 'obvious consequence' of p. (Compare Pap's rule (R1) above.)

Introducing the convention that '$p \vdash q$' is to mean 'q is consequence of p' and that '$p \models q$' is to mean 'q is an *obvious* consequence of p' we may restate these three rules somewhat more precisely and compactly as follows:

(B1) $\Box \sim p \vdash \sim B(x, p)$

(B2) If $q, r \models p$, then $B(x, q), B(x, r) \vdash B(x, p)$

(B3) Never, for all x, $B(x, p) \vdash B(x, q)$, unless $p \models q$.

It is easy to see that, relativized to virtually any system of propositional logic, and given n large enough so that q will be an 'obvious consequence' of '$p \,\&\, (p \rightarrow q)$', and that '$p$' will be an 'obvious consequence' of '$p \,\&\, q$', then all of Pap's axioms can be established on the basis of these three rules. But it is clear also that these rules will not generate the difficulty noted above in the discussion of the unacceptable principle (N2).

A caution about this approach is, however, necessary. The relationship of *obvious consequencehood* is not transitive: p can be an obvious consequence of q and q of r without p being an obvious consequence of r. Hence we must insist that rule (B2) *may occur no more than once* in a valid argument about beliefs. Otherwise we would be back to the unacceptable principle (N2) and the mythology of logical omniscience. For a complex argument from some believed proposition p to a remote conclusion could otherwise be broken up into small, obvious steps, and the belief modality projected across all these by means of repeated applications of (B2).

6. ITERATIONS OF BELIEF

An interesting feature of our rules for belief (as well as Pap's) deserves explicit attention. They require us to reject any interpretation of the concept of belief according to which believing that one believes something is an unfailing guarantee of believing it. For on any such interpretation

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and Other Problems in Epistemic Logic', *Theoria* **32** (1966) 1–13 (see especially p. 3) and 'Are Logical Truths Tautologies?' in *Existenz und Analytizität: 4. Forschungsgespräch des internationalen Forschungszentrums für Grundfragen der Wissenschaften*, ed. by P. Weingartner (Salzburg, 1966).

of belief one can add the rule:

(BB) $\quad B(x, [B(x, p)]) \rightarrow B(x, p).$

This rule – which lays down the seemingly plausible principle that if a person believes that he believes something, then he believes it – would appear, from a strictly intuitive point of view, to be a feasible candidate for acceptance by a logical theory of belief. But note that it at once leads, given rule (B3), to the consequence:

$$B(x, p) \rightarrow p$$

according to which x's believing p has the consequence that p is true, i.e., that only true propositions can be believed. To avoid this absurd consequence in the context of our theory of belief, we must adopt one of two obviously feasible alternatives: (1) We may simply reject the rule (BB), or (2) We may accept (BB) but restrict application of (B3) to such propositions p and q as do not themselves involve the modality $[Bx]$, thus limiting the application of (B3) to modally 'ground floor' propositions (so to speak). Either way, the upshot appears to represent a step in the direction of a viable logical theory for the epistemic modality of belief.

The need for a criterion of adequacy for belief-rules is seen to be all the more pressing and acute because of indications that uncritical reliance upon the concept of 'intuitive acceptability' – which, in the absence of an explicit criterion provides our only guide – will fail us in the construction of a theory of belief statements. Consider again the proposed rule (BB) to the effect that if a person believes that he believes something, then he believes it. This rule seems on first view to be quite as acceptable, from an 'intuitive' standpoint, as any of Pap's. However, in view of (R1), (BB) leads at once to: $(\forall x)\,[B(x, p) \rightarrow p]$. And this contradicts Pap's (R2). Clearly then, we are forced to a choice between (BB) (or R1) on the one hand, and (P2) on the other. In the absence of a criterion of acceptability for proposed rules for the belief relationship we are unable, in this case, and in similar cases which could conceivably arise, to resolve such instances of logical conflict in any manner other than by *fiat*. The possibility of outright logical inconsistency among apparently ('intuitively') acceptable rules of belief of indicates the urgent need for a criterion of acceptability for such rules.

Even without a full-fledged *general* criterion, it would be most useful

to have at least some *partial* criteria that would afford answers to questions of interest, such as the following:

(Q) Under what conditions on the statements '...' and '---' do we have,

(i) $(\ldots \rightarrow ---) \rightarrow (\forall x)\,[\mathrm{B}(x, \ldots) \rightarrow \mathrm{B}(x, ---)]$,

(ii) $(\ldots \leftrightarrow ---) \rightarrow (\forall x)\,[\mathrm{B}(x, \ldots) \leftrightarrow \mathrm{B}(x, ---)]$?

A partial criterion capable only of resolving questions of this sort would be of great usefulness in clarifying the logic of the belief concept and its relationship to the familiar theory of assertory logic.

I am not able here to propose more than a very inconsiderable step in this direction. And this leads to a new topic, the concept of synonymy.

7. BELIEF, SYNONYMY, AND PROPOSITIONS

The belief-relationship which we have been concerned to explicate obtains between a person *and a proposition.* The object of a belief, on this analysis, is *not* a group of words (i.e., a sentence), but a purported fact or state of affairs (i.e., a proposition). In consequence, if '...' and '---' are *synonymous* sentences, i.e., sentences that express the same proposition (convey the same information), then '$\mathrm{B}(x, \ldots)$' and '$\mathrm{B}(x, ---)$' are alternative formulations of the same belief-relationship, i.e., are logically equivalent. To this, it might be objected that it could happen that the person x in question might *fail to recognize* the synonymy of '...' with '---', just as such recognition failures can occur with respect to other logical relationships. It must be granted as being beyond question that this can indeed occur, but that is quite irrelevant. We can, will, and must, in constructing a formal theory of belief postulate a *complete knowledge of the language* (and thus an awareness of all *meaning relationships* obtaining within it) on the part of the person who makes the belief statement.[7] In consequence

[7] This presupposition of a (complete) knowledge of the *language*, i.e., of *meaning* relationships, is, of course, far more plausible and defensible than a presupposition of a (complete) knowledge of all *logical* relationships. Just this consideration justifies acceptance of (C) below in the face of a rejection of an unqualified (Q-ii) above. The distinction between the *source* of the belief statement (its assertor) and its *subject* (the person about whose beliefs the assertion is made) is crucial here. If in the *source's* language, say, 'vixen' and 'female fox' are synonyms, then he has *said the same thing* when he says 'Smith saw a vixen' and 'Smith saw a female fox'.

of this assumption, made in the interests of the formulation of a workable logical theory, all semantical relationships of synonymy must (unlike *logical* relationships) be taken as 'known'.

The following partial criterion of acceptability for belief statements is thus established:

(C) If the statements '...' and '---' are synonymous (i.e., resemble each other in every particular except that one or more occurrences of a term or phrase occurring in '...' have been replaced in the corresponding occurrences in '---' by a synonymous term or phrase), then '$B(x,...)$' and '$B(x,---)$' are logically equivalent. Thus,

$$(...\,S---)\rightarrow(\forall x)\,[B(x,...)\leftrightarrow B(x,---)],$$

where 'S' represents the relationship of synonymy (here among statements).

The range of application of this partial criterion is highly limited, due to the logical strength of the antecedent condition, which requires not merely that '...' and '---' be logically equivalent (as with an unqualified Q(ii) of the foregoing section), but indeed that they be strictly synonymous, i.e., express identically the same proposition.[8]

In taking the belief-relationship to obtain between a person and a *proposition*, rather than a sentence, we take a step that is, it would seem, absolutely essential if our logical theory of belief is to be at all adequate to the belief concept with which we operate in common discourse. Such statements as 'Cataline believed that Cicero was his enemy' neither asserts nor implies anything whatsoever about the actual statements or utterances of the believers in question.

The question of whether a satisfactory theory of belief statements can

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If source wants to claim (say) that 'Smith does not realize that a female fox is a vixen' then he (source) had best say 'Smith does not realize that a female fox is also called a "vixen" '. Otherwise the familiar use-mention puzzles will arise in this context.

[8] It follows by (C) it it is logically possible that a person x can believe '. . .' and yet not believe (or even disbelieve) '---', then '. . .' and '---' cannot be synonymous. This would swiftly lead to the conclusion that no two symbolically (inscriptionally) distinct propositional expressions can by synonymous, were it not for our stipulation of a complete *knowledge of the language*. Just this failure to stipulate a knowledge of the language had led several writers into the anomalous position of a categorical denial of synonymy among inscriptionally distinct expressions. See N. Goodman, 'On Likeness of Meaning', *Analysis* **10** (1949) 1–7. This paper occasioned lively controversy, much of which is cited in Goodman's subsequent paper 'On Some Difference about Meaning', *ibid.*, **13** (1953) 90–96.

be based upon a concept of the belief relationship as obtaining between persons and sentences, inscriptions, utterances, or some such more concrete linguistic entities, rather than propositions has been the subject of intensive debate in recent years.[9] While this discussion has not issued in a decisive termination, it has tended to suggest that the course we have here taken, in holding that the object of a belief is a proposition, is the simplest procedure, and that most adequate to the logical demands of an acceptable analysis.

Of the inscriptional analyses of belief, the most acceptable is that of I. Scheffler, who proposes (in effect) to interpret $B(x, p)$ as 'There is an inscription y which is a That-p inscription (i.e. an inscription representing a sentence which asserts the proposition p), and x has written the inscription y (with assertive intent)'. (For the sake of simplicity of discussion, this analysis ignores the existence of spoken languages, and treats all languages as written.) Yet even this best analysis is open to the most serious objections. In the first place, it only obscures, but does not dispense with a recourse to propositions, since the only imaginable criterion for determining the That-p status of a particular inscription is by referring it to the proposition that p. And even more serious yet is the problem that a That-today-is-a-rainy-day inscription, say in Malay, may *also* be a That-today-is-a-sunny-day inscription, say in Hindustani (or some other language). Thus the assertive commitment of an inscription is, or can be, so ambiguous a matter as to render meaningless any theory of assertion or belief that is based upon it. On balance, the most satisfactory course is to base the logic of belief (and of assertion) upon a propositional analysis, rather than one articulated in terms of sentences, inscriptions, utterances, or the like.

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[9] The starting point of this debate was Alonzo Church's criticism ('On Carnap's Analysis of Statements of Assertion and Belief', *Analysis* **10** (1950) 97–99) of the analysis of belief statements proposed by R. Carnap in *Meaning and Necessity* (Chicago, 1947). Church presents an argument – whose force is, to my mind, conclusive – directed 'against alternative analysis [of statements of assertion and belief] that undertake to do away with propositions in favor of such more concrete things as sentences'. Attempts to answer Church include: I. Scheffler, 'An Inscriptional Approach to Indirect Quotation', *Analysis* **14** (1954) 83–90; H. Putnam 'Synonymity and the Analysis of Belief Sentences', *Analysis* **14** (1954) 114–122. See also A Church, 'Intensional Isomorphism and Identity of Belief', *Philosophical Studies* **5** (1954) 65–73.

8. CONCLUSION

The foregoing discussion of the logical theory of belief-statements has been of a largely negative character, being concerned primarily with the problems and difficulties that confront the development of an adequate theory of this kind. However, certain positive and, I think, useful conclusions do emerge. In concluding, I should like now to survey several of these implications.

First of all, it has become clear that the logical theory of belief statements must be prepared to take account of relationships among statements that are more delicate and more subtle than is requisite in other contexts. It must necessarily draw distinctions over which the ordinary assertory logic can with impunity ride roughshod. This is due to the fact that it is quite possible to be in ignorance of various logical relationships (e.g., entailment, equivalence, etc.) that in fact obtain among believed statements, and so, for example, to believe the axioms of a theory, and yet disbelieve some theorem that follows from them.

This consideration also establishes, in a quite general and fundamental way, the limitations of 'intuitive acceptability' as a sole criterion of acceptability in the construction of a formal axiomatic theory for *any* domain whatsoever. For it is clear that we can be willing to grant the 'intuitive acceptability' of the proposed axioms of the theory (i.e., stand ready to 'believe' them), and yet must nevertheless be prepared for the contingency that these axioms lead to unacceptable consequences, which, due to logical astigmatism, we did not foresee, and which are not 'intuitively acceptable' (indeed they might involve outright inconsistency). Of course we cannot simply dispense with 'intuitive acceptability' as a criterion in our attempts at theory construction. But it is a criterion which must be used only with circumspection and caution.

This points quite generally to the desirability, in the development of any axiomatic theory for the formal logic of some particular concept, of a prior formulation of explicit general criteria of acceptability for assertions within this sphere, thus relieving ourselves of an exclusive dependence upon 'intuitive acceptability' as a sole and exclusive guide. Our scrutiny of one proposed set of axioms for the logical theory of belief statements has served to show that this general desideratum applies in particular, with special force and urgency, to the logic of belief.

Another consideration to have emerged from our discussion is the recognition that it is necessary, in the interests of an effectual distinction between the logic of belief and the logic of assertion, to lay down some proposition (such as Pap's R2) capable of establishing a distinction between *belief* and *fact*, between $B(x, p)$ and p (and incidentally to establish the compatability of $B(x, p)$ with $B(x, q)$ in the face of $p \rightarrow \sim q$). This lends a strong presumption of evidence for the conclusion that the theory of belief statements cannot be articulated without recourse to the logical modalities. An adequate logical theory of belief statements is not, it would seem, to be had unless modal concepts be presupposed as an available tool for its development.

But perhaps our most important lesson has emerged from considering the belief relation in the context of the concept of synonymy. This is the point that there is good reason for holding that the belief relationship obtains between a person and a *proposition*, and that it will not serve the interests of a workable theory of belief to interpret this relationship as standing between a person and some more concrete kind of linguistic entity, such as sentences, utterances, or inscriptions.[10]

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[10] This chapter is a somewhat expanded version of the author's paper of the same title in *Philosophy of Science* 27 (1960) 88–95. It was thus written some years before the appearance in 1962 of Jaakko Hintikka's highly interesting but controversial book, *Knowledge and Belief* (Ithaca, 1962), perhaps the principal defect of which resides in just the point at issue here – viz., in enunciating rules for a logic of belief in the absence of the development of a prior test-criterion for the acceptability of such rules.

CHAPTER VI

MANY-VALUED LOGIC

PART I

HISTORICAL BACKGROUND

1. PREHISTORY

Throughout the orthodox mainsteam of the development of logic in the West, the prevailing view was that every proposition is either true or else false (although which of these is the case may well neither be *necessary* as regards the matter itself nor *determinable* as regards our knowledge of it). This thesis, commonly called the 'law of bivalence' – constituting one key articulation of the 'law of the excluded middle' – was, however, already questioned in antiquity. In Ch. IX of his treatise *On Interpretation (De interpretatione)*, Aristotle discussed of the truth-status of alternatives regarding 'future-contingent' matters, whose occurrence – like that of the sea-battle tomorrow – is not yet determinable by us, and may indeed actually be undetermined. His views on the matter are still disputed, but many commentators, both in antiquity and later, held him to assert that propositions about future contingents, like that asserting the occurrence of the sea-battle, are neither actually true nor actually false, but potentially either, thus having – at least prior to the event – a third, indeterminate truth-status. The acceptance of the principle of bivalence was, in antiquity, closely bound up with the doctrine of determinism. The Epicureans, who were indeterminists, rejected the law of bivalence; the Stoics (and above all Chrysippus) who were rigid determinists, insisted upon it.[1]

In medieval times this problem of the truth-status of future-contingents was much discussed by logicians, both in the Islamic orbit and in Latin Europe.[2] One school of thought classed such propositions as indeterminate

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[1] See N. Rescher, *Studies in the History of Arabic Logic* (Pittsburgh, 1963), pp. 43–44.
[2] *Ibid.*, pp. 43–54.

– i.e., neither true or false. This position did not, however appeal to the most prominent figures on either the Islamic side (e.g., Alfarabi) nor the Christian (e.g., Aquinas), because it involved theological difficulties – how can there be divine foreknowledge if future-contingents statements are neither true nor false? A particularly keen debate about these problems took place at the University of Louvain in the 15th century, with Peter de Rivo as the principal advocate of an 'indeterminate' truth-value.[3] Difficulties about divine foreknowledge quite apart, it is not easy to justify granting to

(1) 'It will rain tomorrow' (asserted on April 12)

a truth-status different from that of

(2) 'It did rain yesterday' (asserted on April 14)

because both make (from temporally distinct perspectives) *precisely the same claim about the facts*, viz., rain on April 13. But be this as it may, the fact is that the idea of a third, indeterminate truth-value at issue in these historic discussions of future contingency provided the impetus and leading idea for Lukasiewicz's development of his three-valued logic.

2. EARLY HISTORY (1870–1914)

The founding fathers of many-valued logic were the Scotsman Hugh MacColl (1837–1909), the American C. S. Peirce (1839–1914), and the Russian N. A. Vasil'ev (1880–1940).[4] MacColl sketched a system of propositional logic in which propositions can take on several distinct 'truth'-values, not only the traditional *true* and *false*, but also the modal values of certainty (necessity), impossibility, and variability (contingency).[5] As examples of bearers of these three truth-values, MacColl exhibited the three statements: '2=2', '3=2', and 'x=2', respectively. Peirce conceived of

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[3] See L. Baudry, *La Querelle des futurs contingents (Louvain, 1465–1475), Textes inédits* (Paris, 1950).

[4] Some have sought to find precursors of three-valued logic in such medieval thinkers as Duns Scotus and especially William of Ockham, who treat of the *propositio neutra* as something distinct from the *propositio vera* and the *propositio falsa*. Thus MICHALSKI (1937) and BOEHNER (1945).

[5] See MACCOLL (1897), and also the discussion by Storrs McCall in the article on MacColl in *The Encyclopedia of Philosophy*, ed. P. Edwards (New York, 1967), vol. 4, pp. 545–546.

the idea of a neutral truth-value in connection with Aristotle's problem of future contingency.[6] Vasil'ev presented a theory of 'non-Aristotelian' logic in which the status of propositions may be either *affirmative, negative,* or *indifferent.*[7] The papers of these precursors foreshadowed the later development of the subject.

3. THE PIONEERING ERA (1920–1932)

The actual inauguration of many-valued logic must be dated from the pioneering papers of Jan Lukasiewicz and Emil L. Post, published in the early 1920's, in which systems of many-valued logic are presented.[8] The first of these papers especially represents a thorough going repudiation of two-valued logic. The mainstream of the development of many-valued logic proceeded on the basis of elaborations of Lukasiewicz's ideas – especially in their formulation in his widely-read paper of 1930[9], where the 3-valued logic was generalized to the many-valued, indeed even infinite-valued case. Some of the relevant ideas were given wide circulation in the influential treatise on *Symbolic Logic* brought out in 1932 by Charles H. Langford and Clarence Irving Lewis.[10] An important variant of the three-valued logic of Lukasiewicz was presented in a 1939 paper by D. A. Bochvar.[11] An important pioneering result was achieved by Mordchaj Wajsberg, who succeeded in 1931 in axiomatizing the three-valued logic of Lukasiewicz by the following set of axioms, together with the rule of *modus ponens*[12]:

(1) $\quad p \to (q \to p)$

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[6] *Collected Papers of Charles Sanders Peirce*, ed. by C. Hartshorne and P. Weiss (Cambridge, Mass., 6 vols., 1931–1935), see especially vol. 4, pp. 12–20, 257–265.

[7] See VASIL'EV (1910) and especially VASIL'EV (1912). Reference should also be made to KLINE (1965). Actually, Aristotle himself is generally held to have assigned a third, neuter truth-value to future-contingent propositions. Consequently logical systems with truth-values over and above T (true) and F (false) should not be characterized as 'non-Aristotelian' but as 'non-Chrysippean' after the Stoic logician Chrysippus, who explicitly insisted that all propositions are either true or else false.

[8] See LUKASIEWICZ (1920) and POST (1921).

[9] See LUKASIEWICZ (1930b).

[10] See LEWIS and LANGFORD (1932), pp. 213–234.

[11] See BOCHVAR (1939).

[12] See WAJSBERG (1931).

(2) $(p \to q) \to [(q \to r) \to (p \to r)]$

(3) $(\sim p \to \sim q) \to (q \to p)$

(4) $[(p \to \sim p) \to p] \to p.$

4. A SURVEY OF RECENT WORK

In the days since its inaugural, pioneering era – say the period 1920–1932 – the theory and applications of many-valued logic have been developed along a great variety of lines. We shall now outline some of the more important of these (for details, the classified index of our bibliography should be consulted).

Many recent writers have recently carried further the study of the three-valued logic of Lukasiewicz and his many-valued generalization of it. Recent papers by C. C. Chang, Helen Rasiowa, and Alan Rose are especially significant in this connection.[13]

Various systems of propositional many-valued logic have been devised apart from the original three-valued system of Lukasiewicz. In this regard, apart from the continuing contributions of Lukasiewicz himself, the joint work of James Barkley Rosser and Atwell R. Turquette is especially important.[14] The systematic survey by Alonzo Church of many-valued truth-tables for propositional logic that are "non-normal" is also of great value in this connection.[15] No doubt the most important application of many-valued truth tables is their proof-theoretic application in demonstrating the independence of axioms. For a detailed exposition of this use, Church's *Introduction to Mathematical Logic* should be consulted.[16]

Certain special features of logical connectives serve to set many-valued systems of logic apart from their two-valued cousin. The need for special operators and connectives in many-valued logics – especially to assure functional completeness – was recognized early in the development. Here the 1930 paper by Jerzy Slupecki is a pioneering contribution.[17] Negation in particular has special features in many-valued logic, due to the absence

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[13] See CHANG (1959), RASIOWA (1950), and ROSE (1950b), (1951d), and (1952).
[14] See ROSSER and TURQUETTE (1952).
[15] See CHURCH (1953).
[16] See CHURCH (1956).
[17] See SLUPECKI (1930).

of the *tertium non datur* principle. The pioneer investigators here were, apart from Lukasiewicz himself, the mathematicians L. E. J. Brouwer[18] and A. N. Kolmogorov[19], and recent contributions include papers by T. Evans and L. Hardy, Eric Foxley, Norman M. Martin, and A. Rose.[20]

The construction of quantification theory for many-valued logic is a relatively late development. The adjustment of quantification theory to fit within the framework of many-valued logic was launched in 1939 in an important paper by J. B. Rosser,[21] and the continuing collaborative work of Rosser and A. R. Turquette is of special importance in this connection.[22] A recent contribution from a new point of departure is made in a 1964 paper by Nicholas Rescher.[23]

The question of the semantic interpretation of many-valued logics is a rather vexed one. Only relatively modest results have been achieved. Papers by Arthur N. Prior and N. Rescher should be consulted for a discussion of problems and a description of the current situation.[24]

The axiomatization of systems of many-valued logic is a matter to which logicians have continued to give a good deal of attention. A detailed examination of the situation is made in the book by J. B. Rosser and A. R. Turquette.[25] A 1961 paper by Andrej Mostowski is a significant subsequent contribution.[26] In an important paper of 1955, Karl Schröter devised means for Gentzenizing systems of many-valued logic (i.e., axiomatizing them by means of sequents).[27] One recent significant result along these lines is the demonstration in 1958 by A. Rose and J. B. Rosser[28] of Lukasiewicz's conjecture that $\mathbf{L}_{\aleph_0}$ (the denumerably many-valued system of Lukasiewicz) is axiomatized by the five axioms:

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[18] See BROUWER (1925) and (1933).
[19] See KOLMOGOROV (1925).
[20] See EVANS and HARDY (1957), FOXLEY (1962), N. M. MARTIN (1950), (1951), (1954), and ROSE (1951a), (1953), (1961), (1965).
[21] See ROSSER (1939).
[22] See ROSSER and TURQUETTE (1948), (1951), and (1952).
[23] See RESCHER (1964).
[24] See PRIOR (1955) and RESCHER (1965).
[25] See ROSSER and TURQUETTE (1952).
[26] See MOSTOWSKI (1961a) and (1961b).
[27] See SCHRÖTER (1955).
[28] See ROSE and ROSSER (1958).

(1)–(3) as above

(4) $[(p \to q) \to q] \to [(q \to p) \to p]$

(5) $[(p \to q) \to (q \to p)] \to (q \to p)$.

together with the rule of *modus ponens*.[29]

There have been various attempts to use many-valued logics for the study of modal logics, and to articulate ideas of modality by means of many-valuedness. For the earlier phases of such efforts a series of papers published by Robert Feys in 1937–1938 should be consulted.[30] In this general context, a 1932 paper by Kurt Gödel is a landmark,[31] for it is there established that no finite characteristic matrix representation can be given for intuitionistic propositional logic. As a consequence, an insuperable cleavage between modal and many-valued logic is made manifest.[32] A recent contribution of some interest because it exploits probabilistic ideas to build a bridge from many-valued logic to modal logic is presented in a 1965 paper by N. Rescher.[33]

Systems of finitely many-valued logic that are not actually truth-functional have only recently begun to be studied. The first systematic effort in this direction is that presented in a 1962 paper by N. Rescher.[34]

The very earliest applications of many-valued logics was to building up the logical groundwork of mathematical intuitionism. Such applications were envisaged as early as 1923 by L. E. J. Brouwer,[35] to whom the key insights in this area must be credited. Now classic papers in this connection are due to Johann von Neumann and Kurt Gödel.[36] Some explanations of various aspects of the connection between many-valued logic and intuitionism have been made in papers by A. Heyting, J. Lukasiewicz, and T. Sugihara.[37]

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[29] That (4) is dispensable was proved (independently) in MEREDITH (1958) and in CHANG (1958).

[30] See FEYS (1937–38).

[31] See GÖDEL (1932) and cf. DUGUNDJI (1940).

[32] For an attempt to breach this gap with an infinite-valued probabilistic (and hence non truth-functional) logic see RESCHER (1963b).

[33] See RESCHER (1965).

[34] See RESCHER (1962).

[35] See BROUWER (1925) and (1933).

[36] See VON NEUMANN (1927) and GÖDEL (1932).

[37] See HEYTING (1956), LUKASIEWICZ (1952), and SUGIHARA (1951).

An important application of many-valued logic to the analysis of the foundational paradoxes in mathematics goes back to a 1939 paper by D. A. Bochvar,[38] who used the third, 'indeterminate' truth-value of a three-valued logic to mean *meaningless* or *undefined*, this truth-value being accorded to paradox-generating propositions.[39] Recently there have also been attempts to apply many-valued logic in the actual development of a paradox-free set theory, an effort to which T. Skolem and especially C. C. Chang have contributed prominently.[40]

The development of syllogistic in the context of many-valued logic has received very little attention. The pioneering work here is a 1938 paper by G. C. Moisil,[41] who has been among the most active contributors to the field generally.[42]

There has been much philosophical interest in many-valued logics. In the main, this has been motivated by preoccupation with the problem of future contingency and the idea of a *neutral* truth value. Starting with Jan Lukasiewicz[43] the idea of a neutral truth-value to reflect the special status of future-contingent propositions has been mooted. The three-valued system of Lukasiewicz has been studied and developed by various writers.[44] In recent years, several papers by A. N. Prior have dealt with various facets of this approach.[45] Particular interest and importance attaches to Prior's applications of infinite-series varieties of many-valued logic in the development of 'tense logics', i.e., logical systems in which relations of propositions to the past, present, and future play a key role in determining their truth status.

Since the late 1920's (highly sporadic) philosophical controversy has raged over the question of whether the several many-valued systems of logic represent logical theories alternative to the orthodox, two-valued logic. The earliest contributors to many-valued logic certainly looked on their efforts as providing such alternatives. Vasil'ev called his system

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38 See BOCHVAR (1939).
39 Compare also MOH SHAW-KWEI (1954).
40 See SKOLEM (1957) and CHANG (1965), where further references are given.
41 See MOISIL (1938).
42 For his full bibliography, see his papers listed in ours.
43 See LUKASIEWICZ (1920).
44 An especially important early development is that of LUKASIEWICZ (1930b).
45 See PRIOR (1953), (1955), and (1957).

'imaginary' and characterized it as *non-Aristotelian*, drawing an explicit analogy with non-Euclidean geometries.[46] Some of the earlier discussions – apart from those on the (already cited) papers by J. Lukasiewicz – are by C.I. Lewis and Paul Weiss.[47] In recent years, the Russian logician A. A. Zinov'ev has been particularly active in this problem area.[48]

Various logicians have attempted to construct algebraic or topological models of many-valued logics. Attempts in the algebraic direction come as early as a 1929 paper by B.A. Bernstein.[49] A topological model was presented by Carl G. Hempel in 1937.[50] The relationships between many-valued logics and lattices have been studied in a series of papers by Alan Rose,[51] who has been among the most active contributors to the field generally.[52]

From the standpoint of mathematical applications, special interest attaches to probabilistic approaches to many-valued logic which began to be explored in the mid-1930's. Here the pioneer was the Polish logician Zygmunt Zawirski,[53] and among the early contributors to this development were Alfred Tarski and Hans Reichenbach.[54] A recent contribution is made in a 1965 paper by N. Rescher.[55]

On the side of physical applications, a handful of writers have applied three-valued logic in the context of indeterminacy-situations of quantum mechanics. The ground was broken in a 1931 paper by Z. Zawirski,[56] but perhaps the most influential discussion is that of a paper published in 1936 by Garett Birkhoff and John von Neumann.[57] More recently, this line of approach has been pursued especially by Hans Reichenbach[58] and

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46 See VASIL'EV (1910) and cf. KLINE (1965), p. 317.
47 See LEWIS (1932a) and WEISS (1933).
48 See especially ZINOV'EV (1963).
49 See BERNSTEIN (1928).
50 See HEMPEL (1937).
51 See ROSE (1950a), (1951d), (1951c), (1958b).
52 For his full bibliography, see his papers listed in ours.
53 See ZAWIRSKI (1934) and (1935).
54 See TARSKI (1935) and REICHENBACH (1935).
55 See RESCHER (1965).
56 See ZAWIRSKI (1931).
57 See BIRKHOFF and VON NEUMANN (1936).
58 See REICHENBACH (1944), (1951), (1952–54), and (1954).

Paulette Destouches-Février,[59] and has been the arena for an ongoing discussion with many contributors.[60]

Great importance attaches to the use of many-valued logics for the formal analysis of electronic circuitry and in switching theory. Here the pioneering work is Claude E. Shannon's paper of 1938.[61] In later years, the Russian mathematician V. I. Shestakov[62] and the Roumanian logician G. C. Moisil[63] have been especially active in this area. (Shestakov's work is based on the 3-valued logic of Bochvar, while Moisil's is based on that of Lukasiewicz.)

In closing this brief historical survey several points seem to be warranted. Many-valued logic is a very new discipline whose origin goes back but a single generation. Not only is it recent, but many-valued logic is also a very 'unfinished' discipline at the present state of its development. It is built up by a great mass of often rather isolated aperçus developed by a considerable army of contributors. All this makes the task of integration, (to which the present work is but a modest contribution) a particularly urgent one.[64]

It was virtually inevitable that many-valued logic would establish itself as an important branch of logic. This is clear from the range and variety of its applications and its connections with problems in other branches of inquiry. Moreover, many-valued logic is of particular interest from the philosophical point of view, because of its deep involvement with such philosophical issues as the question of future contingency, the problem of the logic of indeterminacy in quantum theory, and above all because the multiplicity of diverse systems of many-valued logic poses the question of 'alternative logics' and the whole issue of relativism and conventionalism in logic.

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59 See DESTOUCHES-FÉVRIER (1951).

60 See FEYERABEND (1958), KUZNETZOV (1959), LEVI (1959), MARGENAU (1950), PUTNAM (1957), and TÖRNEBOHM (1957).

61 See SHANNON (1938). A part of Shannon's ideas regarding the analysis of electrical circuitry by means of Boolean algebra were developed independently in V. I. Shestakov's 1938 doctoral dissertation for the University of Moscow.

62 See SHESTAKOV (1946), (1953), (1960).

63 See MOISIL (1956) and (1957).

64 The existing treatises in the field, apart from one brief introduction, namely ACKERMANN (1967), include only TURQUETTE and ROSSER (1952), and ZINOV'EV (1963), both of which deal with a limited set of special problems.

PART II

A SURVEY OF MANY-VALUED LOGIC

1. TWO-VALUED LOGIC

Given a system of propositional logic of the familiar type, it is possible – once the rigorous *syntactical* (structure-oriented, grammatical) development of the system itself has been accomplished – to enter into the area of *semantical* (meaning-oriented) issues. The standard line of approach here is to introduce the two truth-values, T (for *truth*) and F (for *falsity*). One then introduces formal *truth-rules* for the various propositional connectives, spelling out these rules by means of truth-tables. Thus the truth-rules for the usual connectives '$\sim$' (negation), '$\vee$' (disjunction), '&' (conjunction), '$\supset$' (material implication), and '$\equiv$' (material equivalence) are given by the following set of truth-tables:

p	$\sim p$
T	F
F	T

p \ q	$p \& q$		$p \vee q$		$p \supset q$		$p \equiv q$	
	T	F	T	F	T	F	T	F
T	T	F	T	T	T	F	T	F
F	F	F	T	F	T	T	F	T

The content – if not this particular form – of these truth-tables will be entirely familiar to the reader. In the second tabulation, the truth-value for p is given along the left-hand margins, and that for q along the top margin, so that the bottom right-hand entry of the &-table, for example – namely F – tells us that when 'p' takes on the truth-value F (see its column entry), and 'q' does likewise (see its row entry), then '$p \& q$' will be false (i.e., takes on the truth-value F).

It is possible to provide truth-tables of this sort for all the propositional connectives at issue. Thus all of these propositional connectives are – in view of their intended role and meaning – strictly *truth functional.* They are such that, given the truth-values of the components, the truth-value of the resulting compound is always uniquely determined. For example, it is of the very essence of the meaning of '&' that a conjunction of the form '$p \& q$' will be true just exactly when both components are true, and will always be false otherwise.

In the familiar case of the orthodox system of propositional logic we

thus have to do with a two-valued – and strictly truth-functional – system of propositional logic. This system, the *classical* two-valued propositional calculus, we shall designate as the system **C**. It represents the starting-point for the development of many-valued systems of logic.

A (two-valued) *tautology* is a formula that always takes the truth-value T according to the indicated truth-tables, regardless of what truth-values may be assigned to its component propositional variables. For example:

$$p \vee \sim p$$
$$\sim (p \,\&\sim p)$$
$$p \supset p.$$

The objective of an axiomatization of two-valued logic is to give axioms and rules of inference from which all tautologies, but only the tautologies, can be derived as theorems. Such an axiom-system, is said to be *complete* for two-valued logic. Note that it is the semantical basis – and not the axiom-system – that is the starting-point for such an articulation of propositional logic. The truth-tables provide a pre-existing criterion of acceptability (viz., tautologousness) to which the axiom-system must conform at the peril of inadequacy.

2. THE THREE-VALUED LOGIC OF LUKASIEWICZ

A possible step beyond the simplest case of two-valued logic is the introduction of a third, 'intermediate' truth-value I. This was first done by J. Lukasiewicz, who, in discussions beginning with a paper of 1920 motivated his approach in much the manner of his classic 1930 paper:

> I can assume without contradiction that my presence in Warsaw at a certain moment of time next year, e.g., at noon on 21st December, is not settled at the present moment either positively or negatively. It is therefore *possible but not necessary* that I shall be present in Warsaw at the stated time. On this presupposition the statement 'I shall be present in Warsaw at noon on 21st December next year' is neither true nor false at the present moment. For if it were true at the present moment my future presence in Warsaw would have to be necessary, which contradicts the presupposition, and if it were false at the present moment, my future presence in Warsaw would have to be impossible, which again contradicts the presupposition. The statement under consideration is therefore at the present neither true nor false and must have a third value different from 0, or the false, and from 1, or the true. We can indicate this by

'½': it is 'the possible' which goes at a third value with the 'false' and 'the true'. This is the train of thought which gave rise to the three-valued system of propositional logic.[65]

In carrying out this idea, one will introduce corresponding complications into the truth-rules for our propositional connectives. Lukasiewicz' solution to this problem consisted in his putting forward the following truth-tables based on the indicated counterparts to the five connectives considered in Section 1:

p	$\neg p$
T	F
I	I
F	T

p \ q	$p \wedge q$			$p \veebar q$			$p \rightarrow q$			$p \leftrightarrow q$		
	T	I	F	T	I	F	T	I	F	T	I	F
T	T	I	F	T	T	T	T	I	F	T	I	F
I	I	I	F	T	I	I	T	T	I	I	T	I
F	F	F	F	T	I	F	T	T	T	F	I	T

Here again we have a strictly truth-functional system of propositional logic, but now one that is based on three truth-values, not two. We shall designate this system as $\mathbf{L}_3$, short for 'the three-valued logic of Lukasiewicz'.

The guiding principles of the three-valued logic of Lukasiewicz are readily explained.

(1) There are to be three truth-values, T, I, F (so ordered in terms of decreasing 'truthfulness').

(2) The negation of a statement of given truth-values is its 'opposite' in truthfulness.

(3) The truth-value of a conjunction is the falsest (and of a disjunction the truest) of the truth-values of its components.

(4) The truth-value of '$p \rightarrow q$' is the same as that of '$\neg p \veebar q$' *except* that the truth-value corresponding to I→I is set at T (to assure that '$p \rightarrow p$' will invariably assume the truth-value T).

(5) The truth-value of '$p \leftrightarrow q$' is the same as that of '$(p \rightarrow q) \wedge (q \rightarrow p)$'.

With a view to the future-contingency interpretation of the third truth-value I, Lukasiewicz introduced a modal operator of possibility (symbolically $\Diamond$) into his three-valued logic. This is to be subject to the truth-table:

[65] 'Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls', *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* [Classe III] **23** (1930) 51–77 (see pp. 51–52).

p	$\Diamond p$
T	T
I	T
F	F

That is, '$\Diamond p$' is to be true if 'p' is either true or indeterminate, but is false if 'p' is definitely false. Alfred Tarski, early a pupil of Lukasiewicz', remarked that the formula

$$\neg p \rightarrow p$$

will – according to Lukasiewicz's three-valued truth-table – have exactly this same truth-table, and may thus be used as a definition of '$\Diamond p$' within this framework.

It is readily shown that no two-valued tautology will ever take the truth-value F in $\mathbf{L}_3$. But two-valued tautologies can take the truth-value I and thus fail to be tautologous in the three-valued case – since a *tautology* must uniformly take on the truth-value T for all assignments of truth-values to the variables. Thus the 'law of the excluded middle'

$$p \veebar \neg p$$

fails in $\mathbf{L}_3$, because this entire formula will take on the truth-value I when 'p' does so.

It is a noteworthy feature of $\mathbf{L}_3$ that those of its formulas involving only '$\neg$', '$\veebar$', and '$\wedge$' must take on the truth-value I when all of the propositional variables involved do so. This means that no such formulas can be contradictions – i.e., uniformly take on the truth-value F for *all* assignments of truth-values to the variables. Specifically the 'law of contradiction',

$$\neg (p \wedge \neg p)$$

therefore fails to obtain: '$p \wedge \neg p$' takes the truth-value I when 'p' does so. In this regard, $\mathbf{L}_3$ differs from the intuitionistic propositional logic of Brouwer and Heyting, for which the law of excluded middle also fails, but the law of contradiction does not.

3. THE THREE-VALUED SYSTEM OF BOCHVAR

An important variant system of three-valued logic was presented by the

Russian logician D. A. Bochvar in a 1939 paper.[66] Bochvar proposes to construe I as 'undecidable' in the sense of 'having some element of undecidability about it'. On his construction, item (3) of the preceding section (p. 65) came to be altered by changing the truth-table for conjunction to:

p \ q	$p \wedge q$		
	T	I	F
T	T	I	F
I	I	I	I
F	F	I	T

That is to say, a Bochvarian conjuction will automatically take the truth-value I when either of its components takes this truth-value. (The fact that this happens even when one of the components is false, renders the semantics of the situation somewhat problematic.) The other connectives in Bochvar's system are as follows:

(1) $\neg$ is as in the system of Lukasiewicz

(2) $\veebar$ is defined by the definition:

$$p \veebar q = \neg(\neg p \wedge \neg q)$$

(3) $\rightarrow$ is defined by the definition:

$$p \rightarrow q = \neg(p \wedge \neg q)$$

(4) $\leftrightarrow$ is defined by the definition:

$$p \leftrightarrow q = [(p \rightarrow q) \wedge (q \rightarrow p)].$$

The truth-tables for these connectives will thus be as follows:

p	$\neg p$
T	F
I	I
F	T

p \ q	$p \veebar q$			$p \rightarrow q$			$p \leftrightarrow q$		
	T	I	F	T	I	F	T	I	F
T	T	I	T	T	I	F	T	I	F
I	I	I	I	I	I	I	I	I	I
F	T	I	F	T	I	T	F	I	T

We are to think of I not so much as 'intermediate' between truth and falsity but as *paradoxical* or even *meaningless*. We can think of such meaninglessness in terms of what is at issue in the classical semantical paradoxes – i.e., statements which, like 'This sentence is false' we cannot viably class either as true or as false (if true it is false, if false, true). The

[66] See BOCHVAR (1939). The paper is in Russian, but cf. the review by A. Church in *The Journal of Symbolic Logic* **4** (1939) 98–99, and the correction, *ibid.*, **5** (1940) 119. See also BOCHVAR (1943).

effect of these motivating considerations is that the presence of an I-valued conjunct infects the whole conjunct with meaninglessness. The truth-table for conjunction is thus more suggestively presented as:

p \ q	T	F	I
T	T	F	I
F	F	F	I
I	I	I	I

Here I appears in a particularly strong role: its presence in a conjunction – indeed in any context – overpowers the whole, and reduces it to I status.

In elaborating his system, Bochvar makes use of the idea of two distinct modes of assertion:

(1) The ordinary, straightforward, 'internal' assertion of a formula p as simply: p.

(2) The special mode of 'external' assertion of a formula, represented by the special assertion operator A: Ap.

These two modes of assertion are characterized by the truth-tables:

p	Internal assertion p	External assertion Ap
T	T	T
I	I	F
F	F	F

Subject to this conception, distinct 'internal' and 'external' versions can be defined for the various other connectives. Thus:

Connective	Internal Form	External Form[67]
Negation	$\neg p$	$\rightleftharpoons p = \neg(Ap)$
Conjunction	$p \wedge q$	$p \mathbin{\Bbbwedge} q = Ap \wedge Aq$
Disjunction	$p \veebar q$	$p \mathbin{\Bbbvee} q = Ap \veebar Aq$
Implication	$p \rightarrow q$	$p \Rightarrow q = Ap \rightarrow Aq$
Equivalence	$p \leftrightarrow q$	$p \Leftrightarrow q = Ap \leftrightarrow Aq$

[67] Note that to obtain the 'external' formula corresponding to an 'internal' one we simply put an 'A' before every occurrence of a propositional variable in the 'internal' formula.

Thus when it comes to the 'external' form of the connectives, we are, in effect, working simply with the two classical truth-values T and F, with respect to which even the internal 3-valued connectives agree entirely with the classical, 2-valued case. The truth-tables for Bochvar's 'external' connectives are thus as follows:

p	$\neg p$
T	F
I	F
F	T

p \ q	$p \mathbin{\&\!\!\&} q$			$p \vee q$			$p \Rightarrow q$			$p \Leftrightarrow q$		
	T	I	F	T	I	F	T	I	F	T	I	F
T	T	F	F	T	T	T	T	T	F	T	T	F
I	F	F	F	T	T	T	T	T	T	T	T	T
F	F	F	F	T	T	F	T	T	T	F	T	T

It is not difficult to show that this system contains a fragment isomorphic with the classical, two-valued system **C**. For let any classical tautology of **C** be rewritten with all the classical connectives replaced by their 'external' counterparts in **B**. Then the resulting formula must clearly be a tautology of **B**. It is thus an interesting feature of Bochvar's system that it 'includes' (in a certain sense) the whole of the classical, two-valued logic.

The 'external' connectives have certain advantages *vis-à-vis* the 'internal' ones. For example, consider 'external' negation. The formula '$p \mathbin{\&\!\!\&} \neg p$' will invariably come out false, and one of the pair 'p' and '$\neg p$' will also always come out false. By contrast, it is an obvious disadvantage of Lukasiewicz' system that '$p \wedge \neg p$' does not inevitably come out false (specifically, it takes the truth-value I when 'p' does so).

Let us characterize a formula as *tautologous* for the 3-valued system if it uniformly assumes the truth-value T for all values of its arguments and as *contradictory* if it never does so. Note that:

(1) '$p \leftrightarrow p$' is not a tautology in Bochvar's system (it takes the truth-value I when 'p' does so).

(2) '$p \veebar \neg p$' is not a tautology (for the same reason), although '$p \vee \neg p$' is.

(3) '$p \wedge \neg p$' is a contradiction – although its negation '$\neg (p \wedge \neg p)$' is not a tautology (while '$\neg (p \mathbin{\&\!\!\&} \neg p)$' is).

In a 1954 paper,[68] the Chinese logician Moh Shaw-kwei proposed a Bochvarian interpretation for Lukasiewicz's three-valued system $\mathbf{L}_3$. He

[68] See MOH SHAW-KWEI (1954).

proposed that the intermediate truth-value I be construed as 'paradoxical' and be assigned to propositions like 'This statement is false' which will be false if classed as true and true if classed as false. This interpretation requires us to bear in mind a weaker mode of paradoxicality than Bochvar's actual *meaninglessness* since now $I \wedge F$ and $F \wedge I$ are F (and not I, as with Bochvar). Moreover, on Bochvar's interpretation it is clearly warranted to class the truth-value of $p \wedge \neg p$ as I when 'p' takes I, but on Shaw-kwei's approach this step would be less plausible.

4. THE THREE-VALUED SYSTEM OF KLEENE

Yet another way of setting up a system of 3-valued logic was introduced by S. C. Kleene,[69] and we shall designate this system as the system **K**. In Kleene's system, a proposition is to bear the third truth-value I not for ontological but for epistemological reasons: it is not to be excluded that the proposition may in fact be true or false, but it is unknown or undeterminable[70] what its specific truth-status may be.

Kleene introduces his three-valued connectives by the truth-tables:

p	$\neg p$
T	F
I	I
F	T

q \ p	$p \wedge q$			$p \veebar q$			$p \rightarrow q$			$p \leftrightarrow q$		
	T	I	F	T	I	F	T	I	F	T	I	F
T	T	I	F	T	T	T	T	I	F	T	I	F
I	I	I	F	T	I	I	T	I	I	I	I	I
F	F	F	F	T	I	F	T	T	T	F	I	T

Here we have the standard, 'normal' 3-valued truth-tables for negation, conjunction, and disjunction. However, the principal novelty of Kleene's system *vis-à-vis* the system of Lukasiewicz relates to the concept of implication. For in Kleene's system – where $p \rightarrow q = \neg p \veebar q$ – it is not the case that '$p \rightarrow p$' is a tautology, since this formula takes the truth-value I when 'p' does so, nor (*a fortiori*) is '$p \leftrightarrow p$' tautologous.

Kleene motivated the construction of these truth-tables in terms of a

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[69] *Introduction to Metamathematics* (Amsterdam, Princeton; 1952), pp. 332–340. This system was originally presented by Kleene in an earlier paper, 'On a Notation for Ordinal Numbers', *The Journal of Symbolic Logic* **3** (1938) 150–155.

[70] Conceivably (in mathematical applications) 'undeterminable' by certain specific methods, e.g., by an effective algorithm.

mathematical application. He has in mind the case of a mathematical predicate P (i.e., a propositional function) of a variable x ranging over a domain D where 'P(x)' is defined for only a part of this domain. For example, we might have that:

$$\mathrm{P}(x) \quad \text{iff} \quad 1 \leqslant \frac{1}{x} \leqslant 2.$$

Here P(x) will be:

(1) true when x lies inside the range from $\frac{1}{2}$ to 1

(2) undefined or undetermined when $x=0$

(3) false in all other cases (i.e.,: $[(x \neq 0) \,\&\, (x < \frac{1}{2})] \vee [1 < x]$).

Kleene presented his truth-tables to formulate the rules of combination by logical connectives for such propositional functions. Thus he writes:

> From this standpoint, the meaning of $Q \vee R$ is brought out clearly by the statement in words: $Q \vee R$ is true, if Q is true (here nothing is said about R) or if R is true (similarly); false, if Q and R are both false; defined only in these cases (and hence undefined, otherwise.[71])

In addition to these truth-tables for the propositional connectives that he characterizes as 'strong', Kleene also introduces a family of 'weak' connectives, characterized by the feature that the truth-table of such a connective automatically shows the 'output' truth-value I when any one of the 'input' truth-values is I. By this rule, the 'weak' versions of negation and equivalence will be identical with the 'strong' ones, but new versions will be obtained for the other three connectives, as follows:

p \ q	$p \wedge\!\!\wedge q$			$p \vee\!\!\vee q$			$p \Rightarrow q$			$p \Leftrightarrow q$		
	T	I	F	T	I	F	T	I	F	T	I	F
T	T	I	F	T	I	T	T	I	F	T	I	F
I	I	I	I	I	I	I	I	I	I	I	I	I
F	F	I	F	T	I	F	T	I	T	F	I	T

Kleene motivates these truth-tables in terms of arithmetical propositional functions (as before) with respect to which the key issue is that of recursiveness, i.e., of being able to decide the truth-status of the statement by means of an effective calculating procedure. We can think here

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[71] *Introduction to Metamathematics* (*op. cit.*), p. 336.

of a mechanism that is simply *incapable of processing undeterminate (I-assuming) statements*, so that it gives us output value I whenever it receives an input value of I, even if it does so in a context like I-and-F or I-or-T. Here I plays the role of a factor of all-corrupting meaninglessness.

Kleene's 'weak' connectives are identical with Bochvar's 'internal' ones. And this kinship is not surprising, since the motivating considerations of the two approaches are quite similar.

5. MANY-VALUED GENERALIZATIONS OF THE THREE-VALUED LOGIC OF LUKASIEWICZ

Lukasiewicz generalized his three-valued logic to the case of still more truth-values, indeed to infinite-valued systems. Let these truth-values be represented by real numbers from the interval 0 to 1, and let us indicate the truth value of a formula by writing this formula within slashes, so that $/p/$ represents the truth-value of p. Then Lukasiewicz proposed to use truth-tables based on the arithmetical rules:

$$/\neg p/ = 1 - /p/$$
$$/p \veebar q/ = \max[/p/, /q/]$$
$$/p \wedge q/ = \min[/p/, /q/]$$
$$/p \to q/ = \begin{cases} 1 \\ 1 - /p/ + /q/ \end{cases} \text{according as } /p/ \text{ is } \begin{cases} \leqslant /q/ \\ > /q/ \end{cases}.$$

A somewhat variant approach would be to take '$\neg$' and '$\to$' as primitives, subject to the just-indicated principles for constructing truth-tables, and then to define '$\veebar$' and '$\wedge$' as follows:

$$p \veebar q = (p \to q) \to q$$
$$p \wedge q = \neg(\neg p \veebar \neg q).$$

Let us now divide the interval from 0 to 1 by inserting evenly-spaced division-points, for a total of n points ($n \geqslant 2$):

n	Division-Points (= Truth-Values)	
2	$\frac{0}{1}, \frac{1}{1}$	(0, 1)
3	$\frac{0}{2}, \frac{1}{2}, \frac{2}{2}$	(0, $\frac{1}{2}$, 1)

n	Division-Points (= Truth-Values)
4	$\frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}$ $(0, \frac{1}{3}, \frac{2}{3}, 1)$
⋮	⋮
n	$\frac{0}{n-1}, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, \frac{n-1}{n-1}$.

Given the propositional connectives based on the arithmetical rules given in the first paragraph, and taking the members *of this series* as truth-values, we obtain the series $\mathbf{L}_n$ of Lukasiewiczean many-valued logics. It is readily seen that:

(1) $\mathbf{L}_2$ is identical with **C** if we identify 1 with T and 0 with F.

(2) $\mathbf{L}_3$ is identical with the three-valued system of Lukasiewicz, if we identify 1 with T, $\frac{1}{2}$ with I, and 0 with F.

It is therefore clear that the systems of the series will be many-valued generalizations both of the classical 2-valued system **C** and of the Lukasiewicz' 3-valued system $\mathbf{L}_3$. Moreover, we can envisage the possibility of taking *all* the real numbers from the interval 0 to 1 as truth-valued, and will thus also obtain an infinite-valued generalization $\mathbf{L}_\aleph$ of the 3-valued system of Lukasiewicz.

It should be stressed, moreover, that other, radically different systems of infinitely many-valued logics are possible which can still qualify as generalizations of the basic 3-valued logic of Lukasiewicz, $\mathbf{L}_3$. For example, let the value of a proposition be any real number from the interval from -1 to $+1$ (construing -1 as *false*, $+1$ as *true*, and 0 as *neutral*):

$$-1 \leqslant /p/ \leqslant +1.$$

Let us introduce truth-tables for the propositional connectives by the rules:

$$/\neg p/ = -/p/$$
$$/p \wedge q/ = \min[/p/, /q/, /p/ \times /q/].$$

The truth-tables for other propositional connectives can then be constructed on this basis, if these connectives are introduced by definition in terms of the primitives '$\neg$' and '$\wedge$':

$$p \veebar q = \neg(\neg p \wedge \neg q)$$
$$p \rightarrow q = \neg p \veebar q.$$

Moreover, we can introduce a new style of implication $\Rightarrow$ (distinct from $\rightarrow$) by the rule:

$$|p \Rightarrow q| \text{ is } \begin{cases} 1 \\ 0 \text{ according as } |p| - |q| \text{ is } \begin{cases} \leqslant 0 \\ > 0 \text{ but } \leqslant 1 \\ > 1 \end{cases} \\ -1 \end{cases}.$$

Thus while our propositional logic may be ∞-valued, a $\Rightarrow$ implication must always be true, false, or neutral.

Let us for the moment limit our horizon to the special case of three values $+1, 0, -1$. We now obtain, in virtue of the rules laid down for the mode of many-valued logic at issue, the truth-tables:

p	$\neg p$
1	−1
0	0
−1	1

p \ q	$p \wedge q$			$p \veebar q$			$p \rightarrow q$			$p \leftrightarrow q$		
	1	0	−1	1	0	−1	1	0	−1	1	0	−1
1	1	0	−1	1	1	1	1	0	−1	1	0	−1
0	0	0	−1	1	0	0	1	0	0	1	1	0
−1	−1	−1	−1	1	0	−1	1	1	1	1	1	1

Provided, then, that we adopt $\Rightarrow$ (and not $\rightarrow$) as our basic mode of implication, we see that the many-valued family of systems just described represents yet another, distinctly different way of generalizing $\mathbf{L}_3$.

Consider a variant of the preceding approach. Let the truth-values be the totality of *integers* between $-n$ and $+n$ ($n \geqslant 1$). (Note that this totality will always be odd-numbered.) Let the truth-table generating rules be:

$$|\neg p| = -|p|$$
$$|p \wedge q| = \min[|p|, |q|].$$

Defining '$\veebar$' and '$\rightarrow$' as above, we again obtain in the three-valued case the truth-tables:

p	$\neg p$
1	−1
0	0
−1	1

p \ q	$p \wedge q$			$p \veebar q$			$p \rightarrow q$		
	1	0	−1	1	0	−1	1	0	−1
1	1	0	−1	1	1	1	1	0	−1
0	0	0	−1	1	0	0	1	0	0
−1	−1	−1	−1	1	0	−1	1	1	1

So now, introducing $\Rightarrow$ exactly as in the preceding case, we obtain yet another mode of many-valued (but now finite, and always odd-valued!) generalization of $\mathbf{L}_3$.

As these examples show, a great many varieties of many-valued – and indeed infinite-valued – generalizations of the initial 3-valued logic of Lukasiewicz can be constructed.

It should be noted that the question of the *consistency* of logical systems that are constructed in this way – certain truth-values being classed as designated (truth-like), and tautologous propositions, which take these values uniformly, accorded the status of being *accepted* – is not interesting because it is too readily settled in the affirmative. The very manner of construction of the system provides an arithmetical model that guarantees that it will be consistent in that a proposition and its negation can never both be accepted.

6. THE MANY-VALUED SYSTEMS OF POST

All of the finite systems of many-valued logic with which we have dealt to this point have a mode of negation that has the 'mirror-image' feature. It is of interest to consider a system that does not fall into this pattern. In a paper of 1921, Emil L. Post presented a system of finitely many-valued propositional logic based on the m different truth-values for which, for the sake of simplicity, we shall use the first m positive integers 1, 2, 3, ..., n (rather than following Post's own practice of using subscripted t's).[72] Following in the steps of Russell and Whitehead's *Principia Mathematica*, where negation ($\sim$) and disjunction ($\vee$) are taken as primitives, Post introduced m-valued variants of these connectives $\overset{m}{\neg}$ and $\overset{m}{\vee}$. Negation is governed by the truth-table:

p	$\overset{m}{\neg} p$
1	2
2	3
3	4
⋮	⋮
$m-2$	$m-1$
$m-1$	m
m	1

[72] See E. L. Post, 'Introduction to a General Theory of Elementary Propositions', *American Journal of Mathematics* **43** (1921) 163–185. Cf. W. and M. Kneale, *The Development of Logic* (Oxford, 1962) pp. 568–569.

Thus negation becomes, in effect, an operation which effects a cyclic shift in the truth-values, and thus departs from the mode of negation of the standard systems. Disjunction is governed by a truth-table based on the principle (familiar from the $\mathbf{L}_n$ sequence) that the truth-value of a disjunct is simply the larger of the truth-values of its constituents (i.e., the one higher in its subscripted index), so that $/p \overset{m}{\vee} q/ = \min[/p/, /q/]$. The other propositional connectives can then be introduced in terms of these in the usual way, as follows:

$$p \overset{m}{\wedge} q \text{ FOR } \overset{m}{\neg}(\overset{m}{\neg} p \overset{m}{\vee} \overset{m}{\neg} q)$$

$$p \overset{m}{\rightarrow} q \text{ FOR } \overset{m}{\neg} p \overset{m}{\vee} q$$

$$p \overset{m}{\leftrightarrow} q \text{ FOR } (p \overset{m}{\rightarrow} q) \overset{m}{\wedge} (q \overset{m}{\rightarrow} p).$$

Thus we obtain the Postian family of many-valued systems $\mathbf{P}_m$. Actually, Post contemplated a two-parameter family of systems $\mathbf{P}_m^{\mu}$, obtained by treating as truth-like (i.e., designated) the first μ of the m truth-values of $\mathbf{P}_m$.

Specifically, $\mathbf{P}_3$ will be as follows. (We henceforth suppress the m-indication for the connectives, and resume use of $\veebar$.)

p	$\neg p$
1	2
2	3
3	1

p \ q	$p \wedge q$			$p \veebar q$			$p \rightarrow q$			$p \leftrightarrow q$		
	1	2	3	1	2	3	1	2	3	1	2	3
1	3	3	2	1	1	1	1	2	2	3	3	3
2	3	1	3	1	2	2	1	2	3	3	1	2
3	2	3	2	1	2	3	1	1	1	3	2	3

It is clear that while '$\alpha \veebar \neg\alpha$' is not a tautology (i.e., uniformly 1-assuming wff) of $\mathbf{P}_3$, '$\alpha \vee \neg\alpha \vee \neg\neg\alpha$' will be.

Even though this was not done by Post himself, it is possible to set up an infinite-valued Post-style system $\mathbf{P}_{\aleph_0}$ as follows. Truth-values are to be the numbers 1 and 0 and all fractions of the form $(\frac{1}{2})^k$ with integral exponents k:

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \ldots, \tfrac{1}{2^k}, \ldots, 0.$$

The truth-rules for the various connectives go as follows:

$$
\begin{aligned}
/\neg p/ &= \begin{cases} 1 \\ \frac{1}{2} \times /p/ \text{ according as } /p/ \begin{cases} =0 \\ \neq 0 \end{cases} \end{cases} \\
/p \veebar q/ &= \max[/p/, /q/] \\
/p \wedge q/ &= /\neg(\neg p \veebar \neg q)/ \\
/p \rightarrow q/ &= /\neg p \veebar q/ \\
/p \leftrightarrow q/ &= /(p \rightarrow q) \wedge (q \rightarrow p)/.
\end{aligned}
$$

Note: In the present truth-value series the earlier member is the maximum, rather than the minimum as in the series $1, 2, 3, \ldots, n$.

These same truth-rules also yield a Post-style system with a nondenumerable set of truth-values, $\mathbf{P}_{\aleph}$ if we simply let the truth-values be all of the real numbers from the interval from 0 to 1 (inclusive).

An interesting feature of $\mathbf{P}_{\aleph}$ is that there are no tautologies (uniformly 1-assuming wffs). For no formula (wff) of the form $\neg p$ can yield a 0-or-1 output unless there is a 0-or-1 input (in fact, a 0 input), nor can a formula of the form $p \veebar q$ yield a 0-or-1 output unless there is a 0-or-1 input. It is thus easy to show (by mathematical induction on the number of connectives [$\neg$ or $\veebar$] occurring in a formula) that no formula can be tautologous in $\mathbf{P}_{\aleph_0}$ (and *a fortiori* not in $\mathbf{P}_{\aleph}$). On the other hand, it is easy to see that every finite-valued system $\mathbf{P}_n$ will have tautologies – and indeed tautologies in the sequence of 'laws of the excluded n-th':

$\alpha \veebar \neg \alpha$
$\alpha \veebar \neg \alpha \veebar \neg \neg \alpha$
$\alpha \veebar \neg \alpha \veebar \neg \neg \alpha \veebar \neg \neg \neg \alpha$
etc.

So far, the exposition of the system has proceeded as an exercise in purely abstract manipulation that does little to provide a basis for construing the values at issue to be *truth*-values. And the finding of such a construction is not a trivial matter. For there is no way of identifying truth-values t_i=T and t_j=F within the Postian systems $\mathbf{P}_m$ (for finite $m>2$) so that the truth-tables for the propositional connectives become *normal* in the sense of agreeing with the usual two-valued connectives when only T's and F's are involved. (This is clear from the cyclic character of the truth-table for negation.) All the same, Post did manage to present

a semantical interpretation for his nonstandard m-valued logic. Post's interpretation goes essentially as follows:

1. Let the 'propositions' at issue (represented by *capital* letters) be $(m-1)$-tuples of ordinary, two-valued propositions (represented by small letters), subject, however, to the convention that the *true propositions are listed before the false.*
2. Let P assume truth-value i when exactly $i-1$ elements of P are false.
3. Let $\overset{m}{\neg} P$ be formed by replacing the first false element of P by its denial – but if there is no false element, then all are to be denied. (The negation at issue here is the usual propositional negation.)
4. When $P=(p_1, p_2, \ldots, p_{m-1})$ and $Q=(q_1, q_2, \ldots, q_{m-1})$, then
$$P \overset{m}{\vee} Q=(p_1 \vee q_1, p_2 \vee q_2, \ldots, p_{m-1} \vee q_{m-1}).$$
The disjunction on the right-hand side of this equation is the usual propositional disjunction.

On the basis of this interpretation of the connectives, satisfaction of the truth-tables of Post's m-valued propositional logic is guaranteed. The interpretation of the logic in terms of $(m-1)$-tuples of propositions thus provides a means for constructing Post's m-valued system out of the orthodox, two-valued system. Note, however, that this is not a *propositional* interpretation, but one that proceeds in terms of *sets* of propositions. Here greater flexibility is available, but this fact should not block our vision of the nonstandard nature of the interpretation. We now have to do, in effect, not with a many-valued *propositional* logic, but with a many-valued logic of (duly-ordered) *sets of propositions*; and with propositional operators that represent 'negation' and 'disjunction' (etc.) in only a strained or extended sense of those terms.

7. SOME STRUCTURAL FEATURES OF MANY-VALUED LOGICS

Some important abstract, structural features that characterize many (though not necessarily all) of the most familiar systems of many-valued logic deserve to be reckoned with.

(1) The truth-table for a propositional connective that is the many-valued analogue of one of the two-valued connectives will be said to be *normal* if it includes at least one true-analogous truth-value T (which may, however, be designated

by 0 or 1 or n or in some other way) and at least one false-analogous truth-value F (also perhaps differently designated), and this many-valued table *agrees entirely with the standard two-valued one for the connective in* **C** *when only the two truth-values* T *and* F *are involved.*

A many-valued logic may be said to be *normal* (as a whole) if the truth-tables for all of its basic connectives are normal (with respect to one and the same pair of truth-values T, F).

With the exception of the Postian systems, all of the systems of many-valued logic we have considered thus far are normal in this sense.

(2) We shall characterize the many-valued truth-table of a connective as *uniform* if, whenever the T and F headed entries in a given row (or column) both agree in having the same entry, then the entire row (column) uniformly shows this agreeing truth-value. In a uniform truth-table, agreement in the T–F case turns out to be decisive throughout.

Uniformly in this sense may be looked on as a 'forcing principle' to the effect that when in a truth-table, the two extreme entries of a row (column) agree in the T and F cases, this forces agreement throughout this row (column). That is, if it makes no difference in the outcome truth-value whether the input truth-value is T and F, then it will simply not matter at all what the input truth-value is.

A many-valued logic may be called *uniform* (as a whole) if the truth-tables for all of its basic connectives are so.

The (strong) three-valued system **K** of Kleene is uniform in this sense, but the three-valued 'internal' system **B** of Bochvar is not (see his truth-table for $\rightarrow$; but note that the truth-table for his 'external' $\Rightarrow$ is uniform):

p \ q	$p \rightarrow q$			$p \Rightarrow q$		
	T	I	F	T	I	F
T	T	I	F	T	T	F
I	[I]	I	I	T	T	T
F	T	[I]	T	T	T	T

(3) The truth-table for a many-valued connective is *regular in the sense of Kleene* (K-regular) if it never contains T (or F) as an entry in a row (or column) for one of the 'intermediate' values

different from T and F unless this entry T (or F) occurs uniformly throughout its column (or row). In other words, in a K-regular truth-table a classical, definite truth-value (T or F) will occur in a position governed by an intermediate truth-value only if this definiteness is required by the forcing principle of uniformity (i.e., by agreement of the extreme T and F positions).

A many-valued system will be said to be K-*regular* (as a whole) if the truth-tables for all of its basic connectives are so.

The truth-tables for Kleene's strong system of three-valued logic and for that of Bochvar (=Kleene's weak system) have this property, but the truth-tables for $\rightarrow$ and $\leftrightarrow$ in the three-valued logic of Lukasiewicz violate it:

p \ q	$p \rightarrow q$			$p \leftrightarrow q$		
	T	I	F	T	I	F
T	T	I	F	T	I	F
I	T	T	I	I	T	I
F	T	T	T	F	I	T

Note that the second – but not the first – of these tables would become K-regular by changing the central T to I.

Kleene rightly observes that his (strong) system is uniquely determined as the three-valued system that combines three features:

1. It is *normal* in being an extension of two-valued logic, and so agreeing with it when only T's and F's are involved.
2. It is K-*regular*, and
3. It is *the strongest such system* in the sense that its truth-tables have the classical truth-values T or F whenever this is possible for a three-valued system that is both normal and K-regular.

We shall verify this fact only with respect to the truth-table for disjunction. We begin with a truth-table of the following form, the corner entries being determined by the assumption of normality:

p \ q	$p \veebar q$		
	T	I	F
T	T	(1)	T
I	(2)	(3)	(4)
F	T	(5)	F

Entry (1) will be T, since it should be T or F (the truth-table being the strongest), but it cannot be F (by K-regularity). And exactly the same reasoning applies to entry (2). This brings us to the truth-table:

p \ q	$p \veebar q$		
	T	I	F
T	T	T	T
I	T	(3)	(4)
F	T	(5)	F

Now entry (4) cannot be T or F (by K-regularity), and hence is I. And exactly the same reasoning applies to entry (5). And finally, entry (3) cannot now be T or F (by K-regularity). Thus all three remaining entries must be I, and the expected truth table for $\veebar$ is consequently uniquely determined.

In this connection, however, it is also interesting to observe that Kleene's (strong) system is also uniquely characterized among three-valued systems as *the weakest uniform normal system*: weakest in the sense of introducing a nonclassical truth-value (viz., I) whenever possible. This finding characterizes the intended applications of this system better than the preceding one does: whenever the nonclassical indeterminate I enters the scene, the connective yields I as an output value – unless this output truth-value is otherwise fixed by an agreement among the T and F inputs for this case (which would show that the input truth-values are, in a sense, immaterial so that it does not affect matters even if they are 'intermediate' [undecided, unknown, indeterminate, or whatever]).

To make the next point, two further structural features of a wide variety of many-valued systems must be delineated:

(4) The truth-table for a propositional connective in a system of many-valued logic is *strongly uniform* (S-uniform) if it is such that whenever *the same* truth-value occurs at *any* two positions in a certain row (or column) (not necessarily at just the extremes), then all of the intermediate positions of this row (or column) are filled by the same entry.

(5) Let a many-valued logic be based on an 'order of truth' series, $T, I_1, I_2, \ldots, I_n, F$. Then the truth-table for a connective will be said to be *continuous* if every row (or column) that begins with T and ends with F (or the reverse) has the intervening

positions filled with *all the intermediate truth-values in proper order.*

The truth-tables for all the three-valued systems we have considered (Lukasiewicz, Bochvar, Kleene) are strongly uniform (trivially so, being both uniform and 3-valued), as is $\mathbf{L}_4$. Moreover all these three-valued systems are continuous, as is $\mathbf{L}_4$ also.

8. TAUTOLOGOUSNESS AND DESIGNATED TRUTH-VALUES

The concept of a *tautology* is familiar from the two-valued case: a formula that *uniformly takes on the truth-value* T *for any and every assignment of truth-values to its propositional variables.* This concept is readily generalized to apply to many-valued systems of logic. Given a many-valued system of anything like the normal variety, we can classify certain of its truth-values as *designated* – i.e., as representing 'truth-like' truth-values. A formula will then be a *tautology* of the many-valued logic in question – subject to the specified designations of truth-values – if it uniformly takes on a designated truth-value for any and every assignment of truth-values to its propositional variables.

The case of Lukasiewicz' three-valued logic is instructive. Consider again the truth-tables for $\mathbf{L}_3$:

p	$\neg p$
T	F
I	I
F	T

p \ q	$p \wedge q$			$p \veebar q$			$p \rightarrow q$			$p \leftrightarrow q$		
	T	I	F	T	I	F	T	I	F	T	I	F
T	T	I	F	T	T	T	T	I	F	T	I	F
I	I	I	F	T	I	I	T	T	I	I	T	I
F	F	F	F	T	I	F	T	T	T	F	I	T

If we designate T alone, we have the usual tautologies of $\mathbf{L}_3$, including, say, '$p \rightarrow p$' but not '$p \veebar \neg p$' or '$\neg(p \,\&\, \neg p)$'. But if we designate *both* T and I, then *all of the classical tautologies* (i.e., all tautologies of **C**) *will become tautologous.* The following considerations suffice to show this:

> The above truth-tables are such that the introduction of I's into a formula that takes T for some combination of T–F inputs (alone) can never shift the output to F (but only to I). But if a formula is a two-valued tautology then it will always yield the output truth-value T for any combination of the truth-values T and F for its propositional variables. Thus a

two-valued tautology may take on I, but never F, according to these truth-tables.

As such examples show, it makes all the difference in the world for determining the accepted theses (i.e., tautologies) of a many-valued system which truth-values are selected for 'designation'.

The concept of contradictoriness can also be extended to many-valued logics. In the two-valued case, a formula is, of course, a contradiction if it uniformly takes on the truth-value F for any and every assignment of truth-values to its propositional variables. Now in a many-valued system of anything like the normal variety we can classify certain of its truth-values as *antidesignated* – i.e., as representing 'false-like' truth-values. A formula will then be a *contradiction* of the many-valued logic in question – subject to the specified antidesignations of truth-values – if it takes on an antidesignated truth-value for any and every assignment of truth-values to its propositional variables.

The choice of designated and antidesignated truth-values is not to be looked upon as an arbitrary matter. We would obviously want the tautologies (and contradictions) determined by means of the designation specifications to bear some relation to the situation in the two-valued case. For example, we would not want it to happen that there is some truth-value V which (1) is both designated and antidesignated, and moreover (2) there is some formula which uniformly assumes this truth-value. For then this formula would be both a tautology and a contradiction.

The situation with respect to $\mathbf{L}_3$ is again instructive. If we antidesignate F alone, there will be various contradictions, including '$\neg (p \rightarrow p)$' and '$\neg (p \leftrightarrow p)$' and '$\neg [\neg (p \rightarrow p) \rightarrow q]$'. (Note, however, that there can then be no contradiction involving only $\neg$, $\wedge$ and $\veebar$, since the truth-tables at issue are such that any formula involving only these connectives will uniformly yield I as output truth-value when *all* the input truth-values are I. Thus specifically '$p \wedge \neg p$' and '$\neg (p \veebar \neg p)$' will not be contradictions.)

On the other hand, if we antidesignate both F and I, then all of the classical contradictions (i.e., all contradictions of **C**) will become contradictions. (This is established by considerations that parallel the preceding argument that if both T and I are designated in $\mathbf{L}_3$, then all the classical tautologies will result.)

The preceding remark sets the stage for an important point, viz., that in the consideration of some systems of many-valued logic there may be good reason for *letting one and the same truth-value be both designated and antidesignated.* The principle of the classical two-valued case that all (and only) nondesignated truth-values are automatically to be treated as antidesignated does not apply in the many-valued situation, where the set of antidesignated truth-values may set up as being *either smaller or larger* than the set of nondesignated ones.

The classical negation principle that *The negation of a tautology is a contradiction, and vice-versa* must be re-examined in the light of these considerations. For this principle will hold, in general, only if a rather special condition is satisfied:

> That the truth-table for negation has the 'orthodox' feature of taking all designated truth-values into antidesignated ones (and vice-versa).

Once this special condition is satisfied, the operativeness of the classical principle of the negation-correspondence between contradictions and tautologies will be restored in the context of many-valued logic.

9. CONTAINMENT RELATIONSHIPS BETWEEN MANY-VALUED LOGICS

One many-valued system can be said to be 'contained' in another in several distinct senses, three of which are especially important for our purposes. (We must presuppose that the systems at issue are based upon the same propositional connectives, so that the formulas involved are the same throughout.)

(I) The System **X** may 'contain' the system **Y** in the sense that every tautology of **X** (according to its truth-tables and their designated values) is a tautology of Y (under the same conditions). We shall then say that **X** T-*contains* **Y** – i.e., contains **Y** in respect of *tautologies.* In this sense, for example, all of the normal systems are contained in **C**.

(II) The system **X** may 'contain' the system **Y** in the sense that (1) all the truth-values of **Y** are also truth-values of **X**, and (2) if throughout the truth-tables for **X** one suppresses (i.e., erases) all of those rows and columns headed by those truth-

values of **X** that are additional to those of **Y**, then what remains will simply be the truth-tables of **Y**. We shall then say that **X** S-*contains* **Y** – i.e., contains **Y** subject to certain *suppressions*. For example, consider the following four-valued logic:

p	$\neg p$
T	F
I	I
I*	I
F	T

p \ q	$p \wedge q$				$p \veebar q$				$p \rightarrow q$				$p \leftrightarrow q$			
	T	I	I*	F	T	I	I*	F	T	I	I*	F	T	I	I*	F
T	T	I	I*	F	T	T	T	T	T	I	I*	F	T	I	I*	F
I	I	I	I*	F	T	I	I	I	T	I	I*	I	I	I	I*	I
I*	I*	I*	I*	F	T	I	I*	I*	T	I	I	I	I*	I*	I	I
F	F	F	F	F	T	I	I*	F	T	T	T	T	F	I	I	T

As the shadings show, if we suppress the rows and columns correspondingly to I*, the residual system will be Kleene's (strong) system **K**, which is thus S-contained in the indicated 4-valued system.

(III) The system **X** may 'contain' the system **Y** is we obtain **Y** from **X** by identifying each of the truth-values of **X** with one of the truth-values of **Y**, possible collapsing several **X**-truth-values in the process. We shall then say that **X** I-*contains* **Y** – i.e., contains **Y** subject to certain *identifications*. Thus consider the following four-valued logic:

p	$\neg p$
T	F
I	I*
I*	I
F	T

p \ q	$p \wedge q$				$p \veebar q$				$p \rightarrow q$				$p \leftrightarrow q$			
	T	I	I*	F	T	I	I*	F	T	I	I*	F	T	I	I*	F
T	T	I	I*	F	T	T	T	T	T	I	I*	F	T	I	I*	F
I	I	I	I*	F	T	I	I	I	T	I	I*	I*	I	I	I*	I*
I*	I*	I*	I*	F	T	I	I*	I*	T	I	I	I	I*	I*	I	I
F	F	F	F	F	T	I	I*	F	T	T	T	T	F	I*	I	T

Notice that if we establish the following mapping of the truth-values of the four-valued tables T, I, I*, F into the truth-values of a three-valued logic

T into T
I into I
I* into I
F into F

that is to say if we simply identify I = I*, then these truth-tables become:

p	$\neg p$
T	F
I	I
F	T

p \ q	$p \wedge q$			$p \veebar q$			$p \rightarrow q$			$p \leftrightarrow q$		
	T	I	F	T	I	F	T	I	F	T	I	F
T	T	I	F	T	T	T	T	I	F	T	I	F
I	I	I	F	T	I	I	T	I	I	I	I	I
F	F	F	F	T	I	F	T	T	T	F	I	T

That is, subject to the indicated identifications we compress this four-valued system into the 3-valued (strong) system **K** of Kleene, so that this system is seen to I-contain **K**.

It should be observed in general that no many-valued system which – like all of the three-valued logics we have considered – contains a self-negating truth value V for which

$$\neg V = V$$

can possibly I-contain the classical two-valued system **C**, since V cannot then be identified either with T or with F.

It should be noted that:

(1) S-containment implies (reverse) T-containment, but not conversely.

(2) I-containment does not in general imply (reverse) T-containment (though it will do so in special circumstances) nor, conversely, does T-containment imply I-containment.

(3) S-containment does not imply I-containment, nor, conversely, does I-containment imply S-containment.

That S-containment yields (reverse) T-containment is obvious: if for some truth-tables a formula takes a designated truth-value always and in general, then it must continue to do so within the post-suppression truth-tables. That (reverse) T-containment does not entail S-containment can be seen from the fact that smaller many-valued logics can be T-contained in larger ones.

That T-containment does not imply (reverse) I-containment can be seen by the fact that $\mathbf{L}_3$ is T-contained in **C**, but **C** is not I-contained in $\mathbf{L}_3$. That I-containment does not imply (reverse) T-containment can be seen by the example of:

p	$\neg p$
+1	3
+2	3
−3	1

p \ q	$p \wedge q$ 1	2	3	$p \vee q$ 1	2	3
+1	1	2	3	1	1	1
+2	2	2	3	1	2	2
−3	3	3	3	1	2	3

(Here + and − will be used to mark the designated and antidesignated truth-values.) The compression

1 goes into F
2 goes into F
3 goes into T

will yield the (unorthodox) two-valued system

p	$\neg p$
T	F
F	T

p \ q	$p \wedge q$ T	F	$p \vee q$ T	F
T	T	T	F	F
F	T	F	F	T

But '$p \wedge \neg p$' is a tautology of the two-valued system, though not of the 3-valued one. And '$\neg(p \wedge \neg p)$' is a tautology of the 3-valued system, though not of the two-valued one. Thus neither system can T-contain the other.

However, in the special case that the I-containment-generating compression is such that undesignated values never go into designated values, it can be shown that I-containment implies (reverse) T-containment. The argument goes as follows. Assume system **X** I-contains **Y**. Suppose now that some formula that is a **Y**-tautology were not an **X**-tautology. Then there would be a truth-value assignment in **X** for which this formula would take on **X**-undesignated value. But now consider this assignment in **X** and let each of its value-specifications be subjected to the indicated identifications to yield an assignment in **Y**. Then the **Y**-value of the formula must also take on an undesignated value (because undesignated values are never to be mapped into designated ones by identifications). But then this formula will not be a **Y**-tautology, contrary to assumption. Thus we may define a *natural* compression as one which – unlike the 'unnatural' one specified in the last paragraph – will have the features that it:

1. Always takes designated values into designated ones.
2. Never takes undesignated values into designated ones.

3. Always takes antidesignated values into antidesignated ones.
4. Never takes not-antidesignated values into antidesignated ones.

With all such natural compressions, the tautologies and contradictions of the precompressed system must always be included among those of the compressed one. Thus, for example, since the second 4-valued system sketched on page 85 (with the designations +T, I, I*, −F) can be shown by means of a normal compression to I-contain Kleene's system **K** (with the designations +T, I, −F) we know that all the 4-valued tautologies are **K**-tautologies.

That S-containment does not imply I-containment can be seen from the fact that $\mathbf{L}_3$ S-contains **C**, but does not I-contain it. That I-containment does not imply S-containment can be seen from the fact that the just-discussed four-valued system I-contains **K** (as above), but that it is readily seen – from the inspection of its truth-tables – that this system does not S-contain **K**.

10. PRODUCTS OF PLURI-VALUED LOGICS

Suppose two systems of pluri-valued logic, $\mathbf{S}_1$ and $\mathbf{S}_2$, to be given. We can then form a new, many-valued system – to be designated $\mathbf{S}_1 \times \mathbf{S}_2$ – as the product (the 'Cartesian' product) of these two systems in accordance with the following three rules:

(1) The truth-values of the system $\mathbf{S}_1 \times \mathbf{S}_2$ are to be *ordered pairs* $\langle x, y \rangle$ of truth-values, the first of which, x, is a truth-value of $\mathbf{S}_1$ and the second of which, y, is one of $\mathbf{S}_2$.

(2) The truth-value of a proposition is to be $\langle x, y \rangle$ in $\mathbf{S}_1 \times \mathbf{S}_2$ if, and only if its truth-value is x in $\mathbf{S}_1$ and is y in $\mathbf{S}_2$.

(3) Correspondingly, negation ($\neg$) and the primitive binary logical connectives (ϕ an arbitrary such connective) are to be so specified for $\mathbf{S}_1 \times \mathbf{S}_2$ that their truth-tables are governed by the rules:

$$\neg \langle x, y \rangle = \langle \neg x, \neg y \rangle$$
$$\langle x_1, y_1 \rangle \phi \langle x_2, y_2 \rangle = \langle x_1 \phi x_2, y_1 \phi y_2 \rangle .$$

Subject to these principles, the product of two pluri-valued logics is always a well-defined system of many-valued logic.

Let us consider some cases of the working of these ideas. The product of the classical two-valued propositional calculus **C** with itself – that is,

$\mathbf{C} \times \mathbf{C}$ – will be as follows. We shall here reverse our previous practice and write 0 for T and 1 for F. (Note: When only a small number of truth-values are involved, we may omit writing the brackets of $\langle x, y \rangle$.)

p	$\neg p$	p \ q	$p \wedge q$	$p \veebar q$	$p \rightarrow q$	$p \leftrightarrow q$
			00 01 10 11	00 01 10 11	00 01 10 11	00 01 10 11
00	11	00	00 01 10 11	00 00 00 00	00 01 10 11	00 01 10 11
01	10	01	01 01 11 11	00 01 00 01	00 00 10 10	01 00 11 10
10	01	10	10 11 10 11	00 00 10 10	00 01 00 01	10 11 00 01
11	00	11	11 11 11 11	00 01 10 11	00 00 00 00	11 10 01 00

For a second example, consider the product of the classical two-valued system **C** with Lukasiewicz' $\mathbf{L}_3$, leading to the system $\mathbf{C} \times \mathbf{L}_3$:

p	$\neg p$	p \ q	$p \wedge q$	$p \veebar q$	$p \rightarrow q$
			00 01 02 10 11 12	00 01 02 10 11 12	00 01 02 10 11 12
00	12	00	00 01 02 10 11 12	00 00 00 00 00 00	00 01 02 10 11 12
01	11	01	01 01 02 11 11 12	00 01 01 00 01 01	00 00 01 10 10 11
02	10	02	02 02 02 12 12 12	00 01 02 00 01 02	00 00 00 10 10 10
10	02	10	10 11 12 10 11 12	00 00 00 10 10 10	00 01 02 00 01 02
11	01	11	11 11 12 11 11 12	00 01 01 10 11 11	00 00 01 00 00 01
12	00	12	12 12 12 12 12 12	00 01 02 10 11 12	00 00 00 00 00 00

In general, it is clear that the product of the n-valued systems $\mathbf{S}_1$ with an m-valued system $\mathbf{S}_2$ will be a system $\mathbf{S} = \mathbf{S}_1 \times \mathbf{S}_2$ which is $(n \times m)$-valued.

To see how a product-logic of this sort can in at least some cases be given a semantical interpretation, we shall provide an application to chronological logic.[73] Let time be divided into n intervals, beginning with 'the present' interval, I_1:

time →

I_1 I_2 I_3 I_4 ⋯ I_{n-3} I_{n-2} I_{n-1} I_n

Let the propositions p, q, etc., that are at issue be temporally indefinite, so that they can – like 'It is raining today' – be false at one time yet true at another. Then a proposition will take on an n-place truth-value

$$\langle x_1, x_2, \ldots, x_n \rangle$$

[73] This idea is due in its essentials to A. N. Prior. See his *Time and Modality* (Oxford, 1952; second ed., 1962).

where x_i is T or F according as p is true or false in the i-th interval I_i. This temporal construction of many-valued logic represents a perfectly plausible interpretation for the product-logic:

$$\mathbf{C} \times \mathbf{C} \times \cdots \times \mathbf{C} \quad (n \text{ repetitions}).$$

We thus have an illustration of a semantical interpretation of a tense-logical variety for a product logic.

Other interpretations can be provided along essentially the same lines. Thus let the systems K_i be n different axiom-systems (presumably for propositions expressed in one common language – e.g., different systems of Euclidean and Non-Euclidean geometry). Then let the truth-value of p be

$$\langle x_1, x_2, \ldots, x_i, \ldots, x_n \rangle$$

where x_i is T (or F, or I) according as p is provable (or $\neg p$ is provable, or neither) in the system K_i. This provides a perfectly plausible many-valued logic with a readily intelligible and viable semantical interpretation.[74]

We shall introduce the notation $\prod_n(\mathbf{S})$ for the n-fold product of the pluri-valued system S with itself. Thus:

$$\prod_n(\mathbf{S}) = \mathbf{S} \times \mathbf{S} \times \cdots \times \mathbf{S} \quad (n \text{ repetitions}).$$

We adopt also the convention that $\prod_{\aleph_0}(\mathbf{S})$ in the infinite product of S with itself:

$$\prod_{\aleph_0}(\mathbf{S}) = \mathbf{S} \times \mathbf{S} \times \mathbf{S} \times \cdots \quad (\textit{ad infinitum.})$$

These infinite-valued systems can be provided a temporal interpretation along the already indicated lines, time being divided into denumberably many intervals. In just this way A. N. Prior provides[75] the means for interpreting the infinite self-product of a 3-valued logic by letting time start with 'the present' and construing the three-values as follows:

0 = determinately i.e., invariably true

2 = determinately false

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[74] This interpretation is in its essentials due to Alan Rose. See his paper 'Eight-Valued Geometry', *Proceedings of the London Math. Society* [series 3] **2** (1952) 30–44.

[75] *Op. cit.*

1 = future contingent (i.e., neither determinately true not determinately false).[76]

Such a temporal construction provides an interpretation of an infinite-valued system of propositional logic.

The tautologies of a product system bear interesting relations to those of the component systems. To exhibit this, we shall again make use of two abbreviations. If **S** is a many-valued system, then let:

$\mathrm{D}(\mathbf{S})$ = the set of the designated truth-values of **S**
$\mathrm{T}(\mathbf{S})$ = the set of tautologous formulas (wffs) of **S**.

One basic question to be resolved before a cogent consideration of the tautologies of a product system becomes possible is: What are to be the designated truth-values of $\mathbf{S}_1 \times \mathbf{S}_2$? Here two major alternatives lie open:

Policy #1
$\langle x, y\rangle \in \mathrm{D}(\mathbf{S}_1 \times \mathbf{S}_2)$ iff $[x \in \mathrm{D}(\mathbf{S}_1)]$ and $[y \in \mathrm{D}(\mathbf{S}_2)]$
Policy #2
$\langle x, y\rangle \in \mathrm{D}(\mathbf{S}_1 \times \mathbf{S}_2)$ iff $[x \in \mathrm{D}(\mathbf{S}_1)]$ or $[y \in \mathrm{D}(\mathbf{S}_2)]$.

For the sake of notational convenience we shall say that these two policies result in the systems $\mathbf{S}_1 \boxed{\times} \mathbf{S}_2$ and $\mathbf{S}_1 \otimes \mathbf{S}_2$, respectively. A substantial difference between these two systems is readily established, although it is obvious that we shall in general have:

$$\mathrm{T}(\mathbf{S}_1 \boxed{\times} \mathbf{S}_2) \subseteq \mathrm{T}(\mathbf{S}_1 \otimes \mathbf{S}_2).$$

First we shall prove the theorem:

$$\mathrm{T}(\mathbf{S}_1 \boxed{\times} \mathbf{S}_2) = \mathrm{T}(\mathbf{S}_1) \cap \mathrm{T}(\mathbf{S}_2).$$

The demonstration of this result may be divided into two parts as follows:

To begin with, let us assume that $\xi \in \mathrm{T}(\mathbf{S}_1 \boxed{\times} \mathbf{S}_2)$. It will then be the case that for any truth-value $\langle x, y\rangle$, ξ will take on a designated two-place value

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[76] Prior does not notice, however, that this three-valued logic cannot be strictly truth-functional. For if p takes the truth-value 1, then so does $\neg p$. Consequently:
(1) $1 \wedge 1$ is the truth value of $p \wedge p$, and so must be 1
(2) $1 \wedge 1$ is the truth value of $p \wedge \neg p$, and so must be 2
Subject to the specified interpretation of the truth-values, we thus cannot assign to every place in those truth-tables – and specifically to 1 & 1 – one single, unique truth-value.

in $S_1 \boxed{\times} S_2$, and must therefore take on an S_1-designated value for any assignment in S_1 and also an S_2-designated value for any assignment in S_2. This establishes half of our theorem, viz., $T(S_1 \boxed{\times} S_2) \subset T(S_1) \cap T(S_2)$.

Secondly and conversely, assume that both $\xi \in T(S_1)$ and $\xi \in T(S_2)$. Then ξ will always take on a designated value in S_1 and also will always take on a designated value in S_2. But in this circumstance it must also always take on a designated two-place value in $S_1 \boxed{\times} S_2$. This establishes the remaining half of the theorem.

Note that the theorem has the immediate consequence that:

$$T(S \boxed{\times} S) = T(S).$$

We turn now to the consequences of adopting Policy #2. Here we can prove the theorem:

$$T(S_1 \otimes S_2) = T(S_1) \cup T(S_2).$$

First suppose that:

$$\xi \in T(S_1) \cup T(S_2).$$

Then either $\xi \in T(S_1)$ or $\xi \in T(S_2)$ – or both. But if $\xi \in T(S_1)$, then ξ will, for all assignments of S_1 truth-values to its variables, uniformly take on designated value in S_1. But then, by Policy #2, ξ will uniformly take on a designated value in $S_1 \otimes S_2$. Exactly parallel reasoning applies when we assume $\xi \in T(S_2)$. Consequently $\xi \in T(S_1 \otimes S_2)$ in either case, and this fact establishes that:

$$T(S_1) \cup T(S_2) \subseteq T(S_1 \otimes S_2).$$

It remains to prove the converse of this finding, viz.,:

$$T(S_1 \otimes S_2) \subseteq T(S_1) \cup T(S_2).$$

We shall proceed by *reductio ad absurdum*. Suppose that the indicated relationship fails, so that there is some formula ξ such that:

(1) $\xi \in T(S_1 \otimes S_2)$
(2) $\xi \notin T(S_1)$
(3) $\xi \notin T(S_2)$.

By (2) we have it that in S_1 there is some assignment of truth-values to the variables of ξ for which ξ fails to take on an S_1-designated value. Let this assignment be:

$$x_1, x_2, \ldots, x_n.$$

By (3) we have a similar assignment in $\mathbf{S}_2$ for which ξ fails to take on an $\mathbf{S}_2$-designated value, say:

$$y_1, y_2, \ldots, y_n.$$

But now consider the $\mathbf{S}_1 \times \mathbf{S}_2$ assignment:

$$\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle, \ldots, \langle x_n, y_n \rangle.$$

With this assignment ξ will now take on a two-place truth-value undesignated in $\mathbf{S}_1 \otimes \mathbf{S}_2$. Thus $\xi \notin \mathbf{T}(\mathbf{S}_1 \otimes \mathbf{S}_2)$. But this contradicts (1). Q.E.D.

It is also an immediate consequence of the theorem that:

$$\mathbf{T}(\mathbf{S} \otimes \mathbf{S}) = \mathbf{T}(\mathbf{S}) = \mathbf{T}(\mathbf{S} \boxed{\times} \mathbf{S}).$$

In a self-product system, it does not matter for tautologousness which of our two designation policies is adopted. The added designations of Policy #2 cannot add to the stock of tautologies.

It is of interest to observe that $\mathbf{S}_1 \times \mathbf{S}_2$ will S-contain both $\mathbf{S}_1$ and $\mathbf{S}_2$. This can be seen by suppressing all rows and columns of the $\mathbf{S}_1 \times \mathbf{S}_2$ truth-tables whose second (respectively first) place entry (as a two-place truth-value) differs from any one certain value (say T). Moreover $\mathbf{S}_1 \times \mathbf{S}_2$ will also I-contain both $\mathbf{S}_1$ and $\mathbf{S}_2$. This can be seen by identifying all truth-values of $\mathbf{S}_1 \times \mathbf{S}_2$ that have the same first (or respectively second) place entry in the two-place truth-value.

11. THE PURELY ABSTRACT APPROACH TO MANY-VALUED LOGIC

We characterize as 'abstract' that approach to many-valued logic which proceeds on the basis that we are dealing with assignments of values of some *unspecified* kind, which are not necessarily *truth*-values at all. On such an approach, we overlook wholly the relevance of the 'values' at issue to semantical considerations regarding truth and falsity: the assignment of values to propositions is simply viewed as an abstract sort of symbolic game. (But, following custom, we shall continue to speak of 'truth-tables' and 'truth-values' in this context also.) This (perhaps seemingly unfruitful) procedure is of the greatest usefulness in the study of proof-theoretic issues regarding systems of propositional logic.

Its main application is in the presentation of independence proofs in propositional logic. To show that some formula (wff), say ξ, is inde-

pendent of certain axioms $A_1, A_2, \ldots, A_n$ we may present a set of many-valued truth-tables such that:

(1) Each of the axioms has some feature Φ that can be articulated by means of these truth-tables. (This feature Φ may, for example, be 'uniformly taking the truth-value V for all assignments of truth-values to the propositional variables involved', or 'never taking the truth-value V for any assignment of truth-values to the propositional variables involved' or 'always taking one of the truth-values $V_1, V_2, \ldots, V_n$ for any assignment of truth-values to the propositional variables involved'.)

(2) The rules of inference are feature-preserving (i.e., Φ-preserving) in the sense that when applied to formulas that are feature-possessing they lead to a result-formula that is feature-possessing.

(3) The formula ξ lacks the feature Φ.

In such circumstances it is clear that ξ cannot be derived from the axioms. For a proof of ξ would be a list of formulas:

$$X_1$$
$$X_2$$
$$\vdots$$
$$X_n = \xi .$$

All of these formulas up to some X_j are axioms, and all formulas after X_j are obtained from preceding formulas by the rules of inference. But if the axioms are feature-possessing, and the rules of inference feature-preserving, then *every* formula in the list for a valid proof must be feature-possessing. Then we could never arrive at the feature-lacking formula ξ, thus demonstrating its unprovability from – i.e., independence of – the axioms at issue.

As an example of this technique, consider the following standard set of axioms for the classical propositional calculus[77] (based on the rules of substitution and *modus ponens* as rules of inference):[78]

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[77] These are the axioms for propositional logic given in B. Russell and A. N. Whitehead, *Principia Mathematica*, vol. 1 (Cambridge, 1910).

[78] Here '$\vee$' and '$\sim$' are taken as primitive, '$P \supset Q$' defined as '$\sim P \vee Q$' and '$P \& Q$' as '$\sim (\sim P \vee \sim Q)$'.

(A1) $(p \vee p) \supset p$
(A2) $q \supset (p \vee q)$
(A3) $(p \vee q) \supset (q \vee p)$
(A4) $[p \vee (q \vee r)] \supset [q \vee (p \vee r)]$
(A5) $(q \supset r) \supset [(p \vee q) \supset (p \vee r)]$.

Consider now the following set of three-valued truth-tables (based on regarding $\neg$, $\wedge$, $\veebar$, $\rightarrow$, and $\leftrightarrow$ as the many-valued counterparts to the classical $\sim$, &, $\vee$, $\supset$, and $\leftrightarrow$, respectively):

p	$\neg p$
0	2
1	1
2	0

p \ q	$p \wedge q$ 0 1 2	$p \veebar q$ 0 1 2	$p \rightarrow q$ 0 1 2	$p \leftrightarrow q$ 0 1 2
0	0 1 2	0 0 0	0 1 2	0 1 2
1	1 2 2	0 0 1	0 0 1	1 0 1
2	2 2 2	0 1 2	0 0 0	2 1 0

With regard to these, we observe that these truth-tables are such that: (1) they are in accord with the usual definitions of '$\wedge$', '$\rightarrow$', and '$\leftrightarrow$' in terms of the primitives '$\veebar$' and '$\neg$', with 0 construed as truth and 2 as falsity, (2) the rule of substitution obtains – as it must in any many-valued system – since a substitution cannot introduce new truth-values over and above the initial ones and, (3) the rule of *modus ponens* obtains since if an implication and its antecedent alone both have the truth-value 0, the consequent must also have the truth-value 0.

Returning to the specified axioms, it is easily verified that all of these axioms except for axiom (A1) will assume the truth-value 0 *identically* (for all assignments of truth-values to their propositional components). However, axiom (A1) does not have this 0-assuming feature, for when 'p' takes on 1, then '$p \veebar p$' takes on 0, and so '$(p \veebar p) \rightarrow p$' assumes the value 1, and not 0. However, since all the other axioms are uniformly 0-assuming, and since the rules of inference will preserve this feature, this proves that *axiom* (A1) *cannot possibly be derived from the rest and so is independent of them.* This significant meta-systematic fact is readily brought out by means of our 'truth-table' considerations.

The important thing to notice about these tables is that we cannot really make semantical sense of the '*truth*'-values at issue. We cannot construe 1 as some analogue to *truth* or to *falsity* (for then '$\neg p$' could not take on 1 when 'p' takes on 1). Moreover, according to the truth-table for '$\leftrightarrow$', we have it that both $(1 \leftrightarrow 0) = 1$ and $(1 \leftrightarrow 2) = 1$, which sug-

gests that 1 can be neither true-like nor false-like. In sum, the values of 'truth'-tables at issue do not admit a semantical interpretation as representing '*truth*'-values.

This 'abstract' use of many-valued truth-tables in the construction of meta-systematic independence-proofs is one of their main uses in symbolic logic. It is, however, a very specialized use of such systems, and in general the systems involved qualify as many-valued *logics* only by courtesy. Their 'truth-values' have in fact cut loose from the moorings of a semantical construction in truth-relevant terms, and correspondingly the assignment of such values to propositions has little bearing (indeed no direct bearing) on the characterization of inferences and processes of reasoning. Consequently such technical constructs can be characterized as 'systems of logic' only by metonymy. To say this is not to impugn the possible interest of such systems from a certain point of view. The algebraic analysis of formal properties of multivalent propositional systems may well be of considerable value in its own right. But it would be misleading to characterize such an abstract mechanism as presenting a 'logic'. To have what deserves to be called 'many-valued *logic*' some semantical link of the 'truth'-values at issue to the standard conception of truth-falsity must be present. This kind need not necessarily be with the categorical true-false distinction, but could be to truth and falsity *relative to assumptions*, thus invoking the concept of inferential validity. But *some* involvement with truth – be it categorical or hypothetical – is essential to make the 'truth'-values of a many-valued system into *truth*-values, thus establishing the semantical link which alone can make the many-valued system into a *logic*.

12. DIFFICULTIES IN THE SEMANTICAL INTERPRETATION OF MANY-VALUED LOGICS AND ONE POSSIBLE MODE OF RESOLUTION

It is necessary to consider the principal alternatives for the semantical interpretation of the (non-standard) truth-values of many-valued logic. An examination of the literature of the subject shows that the various major possibilities here are as follows.

(I) One possible alternative is to let the truth-values be assigned not to individual propositions but to ordered *sets* of propo-

sitions, and then to reinterpret such assignments in terms of allotment of the classical truth-values T, F to the propositions comprised in these sets. (E. L. Post.)

(II) Another approach is to consider truth-value assignments not to definite but to *variable propositions* – i.e., propositional functions of a parameter. The main possibility that has been explored in this direction is that of a *temporal* parameter. (A. N. Prior.)

(III) Another approach is to let the truth-values reflect the *modal* (or the ontological) status of propositions within some such schematism as:

necessarily false, actually false, indeterminate, actually true, necessarily true

By aggregation to three cases, one arrives at the traditional 3-valued case: true, indeterminate, false. (J. Lukasiewicz.)

(IV) A further possibility is to let the truth-values reflect the *epistemic* status of propositions, within some such schematism as:

- known to be or demonstrably (provably) false
- neither known to be (provable as) true or known to be (provable as) false
- known to be or demonstrably (provably) true

In this way one can, for example, attempt a many-valued treatment of intuitionistic logic. (J. B. Rosser.)

(V) Another approach is to let the truth-values reflect the semantical status of propositions within the schematism: true, meaningless, false. (Bochvar.)

Let us examine the possibilities along these lines in somewhat greater detail.

The task of providing a plausible semantical interpretation for three-valued logic is a difficult one. For consider the set of (incomplete) truth-tables given at the top of the next page. These truth-tables agree (as is customary) with the usual, two-valued truth-tables whenever only T's and F's alone are involved. But how is the rest of the tables to be

p	$\neg p$
T	F
I	I
F	T

p \ q	$p \wedge q$ T	I	F	$p \veebar q$ T	I	F
T	T		F	T		T
I		X				
F	F		F	T		F

filled in? Let us concentrate for the moment on the entry X; What is its value to be, T, I, or F? Here the following two lines of thought can be brought to bear:

1. Obviously we must have it that $/p \wedge p/ = /p/$. Thus when $/p/ = \mathrm{I}$, then also $/p \wedge p/ = \mathrm{I}$. Thus we must have it – by this line of thought – that X, the truth-value corresponding to the case $\mathrm{I} \wedge \mathrm{I}$, must be I.
2. Obviously $/p \wedge \neg p/$ must be F, regardless of what $/p/$ may be. But when $/p/ = \mathrm{I}$ we also have it that $/\neg p/ = \mathrm{I}$. Now since $/p \wedge \neg p/ = \mathrm{F}$, by our initial principle we must have it – on this line of thought – that X, the truth-value corresponding to the case $\mathrm{I} \wedge \mathrm{I}$, must be F.

This pair of findings poses a dilemma for the issue of semantical interpretation: there *just is no way* at all at arriving at a unproblematically satisfactory specification of the value X within a truth-functional framework.

This shortcoming of three-valued logic inheres in the fact that – on any half-way plausible semantical basis – there will here be a self-negating truth-value, that is, a truth-value V such that $\neg \mathrm{V} = \mathrm{V}$. The natural step to overcome this obstacle is to proceed to a four-valued system. Let us see what can be done along these lines.

It would appear on first thought that the most plausible and tempting possibility of interpretating the truth-values of a four-valued system of propositional logic would be along some such lines as the following. (We shall call the truth-values of the system simply by the numbers 1, 2, 3, and 4.)

Truth-Value	Interpretation I	Interpretation II
1	necessarily true	true
2	contingently true	probably true
3	contingently false	probably false
4	necessarily false	false

It is plain that both 1 and 2 must be taken as *designated* truth-values in

this scheme – i.e., 'truth-like' truth-values which a formula can assume without compromising its status as a tautology. For we wish to keep to the customary meaning of a 'tautology' (i.e., a propositional scheme always taking designated truth-values) as a scheme that is 'uniformly *true* for every truth-value assignment to its constituents'.

In both instances, negation would be characterized by the familiar truth-table:

p	$\neg p$
1	4
2	3
3	2
4	1

Along these lines we might think to obtain a four-valued system discussed by A. N. Prior.[79] For let the four truth-values 1–4 be so construed that their assignment to a proposition has the following significance:

1: true and purely mathematical
2: true but involving reference to facts
3: false but involving reference to facts
4: false and purely mathematical.

The propositions at issue are thus classified according to whether they are 'pure', i.e., strictly abstract or mathematical in subject matter, or 'impure' in involving a reference to facts about natural objects. Such 'impurity' infects any proposition into which it enters conjunctively: 'The cat is on the mat and $2+2=5$' thus automatically takes the truth-value 3 (despite the logical falsity of its second conjunct). Then we might think to have the following truth-table for conjunction in this case:

p \ q	$p \wedge q$			
	1	2	3	4
1	1	2	3	4
2	2	2	3	3
3	3	3	3	3
4	4	3	3	4

Unfortunately however, the same vitiating difficulty arises alike with

[79] See his article 'Logic, Many-Valued' in *The Encyclopedia of Philosophy*, ed. P. Edwards (New York, 1967), vol. 5, pp. 1–5 (see p. 4).

all these cases with regard to the matrix for conjunction. Consider what entry is to correspond to $2 \wedge 3$. In Interpretation II, it is obvious that this cannot be specified as simply 3, since, in the instance of '$p \wedge \neg p$', 4 would clearly be the appropriate entry. Again, in Interpretation I, the entry corresponding to $2 \wedge 3$ will in general be 3, but will have to be 4 if – again as in the case of '$p \wedge \neg p$' – the conjuncts are mutually exclusive. The root difficulty is that in the case of either interpretation it is impossible to carry through a semantically adequate truth-functional specification of a matrix for conjunction.

The approach to be presented is based on a modification of Interpretation I which – by an appropriate but minor change in the intended meaning of the truth-values – *removes* the possibility that conjunctions corresponding to the truth-value compounds $2 \wedge 3$ or $3 \wedge 2$ can possibly be self-consistent, i.e., can yield a value of 3 as well as 4.

Let it be supposed that there are just two possible alternatives (i.e., incompatible states of affairs), the *actual* state X, and the *possible alternative* state Y. To any proposition whatever, we will assign the truth-value 1, 2, 3, or 4 according as it is:

(1) true in X and in Y (i.e., is necessarily true).

(2) true in X but not in Y (i.e., is actually but not necessarily true).[80]

(3) false in X but true in Y (i.e., is actually but not necessarily false).

(4) false in both X and in Y (i.e., is necessarily false).

This interpretation of the truth-values[81] clearly gives rise to the following truth-tables for negation and conjunction:

p	$\neg p$
1	4
2	3
3	2
4	1

p \ q	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

$p \wedge q$

[80] Both 1 and 2 will now of course be *designated* truth-values.

[81] This approach has been evolved, albeit with crucial modifications, from a proposal put forward by A. N. Prior in a paper on 'Many-Valued and Modal Systems: An Intuitive Approach', *The Philosophical Review* **64** (1955) 626–630.

This interpretation of our system of four-valued logic is perfectly viable.

We note that this characterization of negation and conjunction in fact coincides with one of the four-valued truth-tables ('matrixes') originally proposed by C. I. Lewis.[82] We may note further that this system of four-valued logic is the system $\mathbf{C} \times \mathbf{C}$.

A very similar interpretation for this same system is provided by examining propositions which can be true (or false) in two different systems S_1 and S_2 – say in two systems of logic or in two different systems of geometry. Then the four truth-values could be interpreted as

1: true in S_1 and S_2
2: true in S_1 but not in S_2
3: false in S_1 but not in S_2
4: false in both S_1 and S_2.

Along just these same lines, Alan Rose constructs an 8-valued system with geometric propositions – the values indicating their truth-status in the three systems of Euclidean, Riemannian, or Lobatchewskian geometry.[83]

Even after our proposed interpretation of the four truth-values is given, there still remain several distinct, more or less plausible and 'natural' truth-value characterizations for the modality of possibility ($\Diamond$), and then derivatively for a strict implication relationship defined in the familiar way:

$$p \rightarrow q \text{ FOR } \neg \Diamond (p \wedge \neg q)$$

The available possibilities seem to be pretty well exhausted by the following list:

Alternatives for $\Diamond p$

p	(A)	(B)	(C)	(D)	(E)
1	1	1	2	1	1
2	1	2	2	1	2
3	1	2	2	1	2
4	4	4	4	3	3

Here the adoption of (A) leads to a system of many-valued truth-tables

[82] C. I. Lewis and C. H. Langford, *Symbolic Logic* (New York, 1932; reprinted 1959). See Appendix II. All references here to 'Lewis and Langford' are to this appendix which was, however, written by Lewis.

[83] See ROSE (1952).

coinciding with Lewis' 'Group III'.[84] Alternative (C) leads to his 'Group IV' and (D) leads to 'Group I'. Cases (B) and (E) are not considered by Lewis. Contrariwise, Lewis' 'Group II' and 'Group V' are incompatible with our proposed interpretation of the truth-values.[85] The 'Group III' system based on (A) has the feature that it renders tautologous all of the theses of Lewis' modal system S5, if 1 is taken to be the only designated truth-value. But these truth-tables will also verify (i.e., render tautologous) certain formulas that are not S5-tautologies. (The tautologousness of these formulas reflects the fiction of there being only two possible alternative states of affairs.) Consequently, this $\mathbf{C} \times \mathbf{C}$ variant system will not be *characteristic* for S5, since it does not verify *all and only* the theses of S5.

We would, however, obtain a set of truth-tables characteristic for S5 if we employed an infinity of truth-values, each of which is an infinite sequence of 0's and 1's. Each place in this sequence represents one possible world (or – alternatively – one period in the history of the actual world), a proposition taking 0 or 1 according as it is true or false, respectively, in that possible world (or state). The truth-values for compound propositions are to be worked out place by place according to the usual two-valued rules, and '$\Diamond p$' is to be true iff the infinite sequence truth-value for 'p' contains at least one 0. (The only designated truth-value is the series of 0's only, and a formula is a tautology if it invariably assumes a (i.e., the) designated truth-value, regardless of the truth-values of its constituents.)

It would appear that, in terms of our semantical approach to the interpretation of the truth-values, alternative (A) – and thus 'Group III' is the most suitable choice of a matrix for the possibility-modality. It is based on a quite plausible construction of the notion of possibility, the convention that: *A proposition asserting the possibility of some thesis is necessarily true if true, necessarily false if false, and true just in case the thesis at issue is not necessarily false.* None of the other alternatives (B) – (E)

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[84] Since all the theorems of Lewis' S5 are tautologies of system (A) = 'Group III', we can claim that our intuitive interpretation of the four-valued truth-tables – like that of Prior mentioned in footnote 3 above – furnishes derivatively an intuitive interpretation of this system of modal logic.

[85] The system of alternative (B) is, as its characterization makes evident, intermediate between the systems of alternatives (A) and (C) – that is, between Lewis' 'Group III' and 'Group IV'. This system may well repay further study. Suffice it here to remark that it resembles 'Group IV' in that neither of Lewis' axioms A7 ($\sim \Diamond p \rightarrow \sim p$) nor B7 (*modus ponens*) are forthcoming as tautologies.

embodies a set of principles that can readily be defended – as alternative (A) can be – on the basis of traditional conceptions of the matter. Still, the other alternatives are certainly not wholly devoid of claims to consideration.

In general, the result of this discussion may thus be summarized by saying that it has presented a semantical approach to the intuitive interpretation of four-valued logic, and that this mechanism makes possible the interpretation of three of the classical 'Groups' of many-valued logic originally proposed by C. I. Lewis, as well as several cognate systems.[86]

13. VARIETIES OF NEGATION IN MANY-VALUED LOGIC

In a many-valued logic, various types of 'negation' are always possible. We may say that a (monadic) propositional operator N represents a mode of negation if it can never, under any circumstances, happen that the members of the pair p, Np are both true, or are both false (i.e., both never take T and both never take F – or whatever other truth-values represent truth and falsity). We regard this requirement as representing a minimal requisite (necessary condition) in qualifying a monadic propositional connective as a negation operator. This requirement not only preserves an evident point of kinship with the ordinary (two-valued) situation, but assures other points of orthodoxy as well, for example:

If a conjunction ($\wedge$) takes the 'falsest' and a disjunction ($\veebar$) the 'truest' truth-value then:

$p \wedge \mathrm{N}p$ can never be T

$p \veebar \mathrm{N}p$ can never be F.

Thus for the sake of illustration consider again $\mathbf{L}_3$, the 3-valued logic of Lukasiewicz:[87]

p	$\neg p$
T	F
I	I
F	T

p \ q	$p \wedge q$			$p \vee q$			$p \rightarrow q$			$p \leftrightarrow q$		
	T	I	F	T	I	F	T	I	F	T	I	F
T	T	I	F	T	T	T	T	I	F	T	I	F
I	I	I	F	T	I	I	T	T	I	I	T	I
F	F	F	F	T	I	F	T	T	T	F	I	T

[86] The present section draws upon the writer's paper 'An Intuitive Interpretation of Systems of Four-Valued Logic', *Notre Dame Journal of Formal Logic* **6** (1965) 154–156.

[87] Actually, the ensuing observations apply equally to the systems of Bochvar (**B**) and Kleene (**K**).

It would surely not be plausible to subject negation to the requisite that either

1. At least one member of the pair p, Np must be true

or

2. At least one member of the pair p, Np must be false.

For this condition would exile from among the modes of negation all the traditional three-valued logics which (like those of Lukasiewicz, Bochvar, and Kleene) are based on the truth-tables

p	$\neg p$
T	F
I	I
F	T

A characterization of (many-valued) 'negation' that disqualified such historically significant cases would not be acceptable.

We may now introduce several styles of "negation" (in the specified sense) over and above that of Lukasiewicz ($\neg$):

p	$\dashv p$	$-p$
T	F	F
I	F	T
F	T	T

Both of these modes of negation (as well as $\neg$) are 'normal' in the sense of conforming to the truth-table for negation in the two-valued case (when only T's and F's are involved). These three of course exhaust the possibilities for normal negation in three-valued logic. If we drop this requirement of normalcy, still other modes of three-valued negation become possible, such as:

p	$\sqsubset p$	$\approx p$	$\simeq p$	$\sqsupset p$	$=p$
T	I	I	F	I	I
I	T	I	T	I	F
F	I	T	I	I	I

Given any mode of conjunction ($\wedge\!\!\!\wedge$), we can measure the strength of a mode of negation (with respect to this mode of conjunction) by the proportion of cases in which '$p \wedge\!\!\!\wedge$Np' takes the truth-value F. (Note that

this cannot in the present case, be less than $\frac{1}{3}$, since – by the specific mode of conjunction $\wedge$ at issue – the conjunction takes the truth-value F when $/p/ = \mathrm{F}$, regardless of what we do about $/\mathrm{N}p/$.) Thus we have:

N	strength
$\neg$	$\frac{2}{3}$
$\dashv$	$\frac{2}{3}$
$-$	$\frac{2}{3}$
$\vdash$	$\frac{1}{3}$
$\approx$	$\frac{2}{3}$
$\simeq$	$\frac{2}{3}$
$\models$	$\frac{1}{3}$
$=$	$\frac{1}{3}$

Of course, some of the weaker modes of negation will appear in a very strange light when one contrast the principles (acceptable theses) that govern them with those obtaining of orthodox negation. Consider, for example, the thesis:

$$(p \leftrightarrow \mathrm{N}p) \leftrightarrow \mathrm{N}(p \leftrightarrow \mathrm{N}p).$$

This thesis is clearly a distinctly odd one. Yet is a logical truth (i.e., is tautologous, uniformly true) if either $\models$ or $=$ is taken to be the mode of negation (N) in question. However, this sort of anomaly cannot arise when all of the propositional connectives are *normal*, and so conform to the 2-valued case whenever only T's and F's are involved. This is so because the many-valued tautologies must then form a subset of the two-valued ones, so that the many-valued system is a fragment of the classical two-valued propositional calculus.

Let us call a many-valued mode of negation *natural* if there is no self-negating truth-value, i.e., no truth-value V such that:

$$\neg \mathrm{V} = \mathrm{V}.$$

Moreover, we may speak of a negation as K-*regular* if only the definite truth-values (T and F) assure a definite truth-value for their negation. It is then clear that

(1) No three-valued negation can be both natural and K-*regular*.

(2) The only normal four-valued negation that is natural and K-regular is:

p	$\neg p$
T	F
I_1	I_2
I_2	I_1
F	T

That is, the conditions of normalcy, naturalness, and K-regularity uniquely determine this mode of negation.

(3) Whenever more than four truth-values are involved, various different modes of negation that are normal, natural, and regular can be devised.

Finally, let us characterize a mode of negation as having *the mirror-image property* if the truth-values at issue can be arranged in an 'order of truth (or falsity)' and the truth-table for negation simply assigns the opposite truth-value:

p	$\neg p$
T	F
I_1	I_k
I_2	I_{k-1}
⋮	⋮
I_{k-1}	I_2
I_k	I_1
F	T

A negation-operator that has this mirror-image property will automatically be normal (i.e., in agreement with the two-valued case) and K-regular. It will moreover also have to be natural whenever an even number of truth-values are involved (though never otherwise).

The general situation regarding negation in many-valued logic is best approached with a view to the distinction between designated and antidesignated truth-values introduced in Sect. 8 above. A mode of negation may be characterized as *quasi-natural* (with respect to a fixed group of designation-specifications) if the negation of a designated truth-value is always antidesignated, and conversely. This feature at least assumes that the negation of a tautology will be a contradiction, and conversely.

A. A. Zinov'ev has made the following claim:

In two-valued logic the negation of truth is falsity, and the negation of falsity, truth. In many-valued logic this should be preserved in some form: if i is the truth value corresponding to truth, and k the one corresponding to falsity, then $Ni = k$ and $Nk = i$ should be the case. Without this condition every comparison becomes meaningless.[88]

On this contention the systems with a non-normal mode of negation would be ruled out altogether. But this is too drastic a step. The workings of systems with unorthodox modes of 'negation' can be perfectly coherent and intelligible. Viable semantic interpretations can be provided for them (as we shall see below in the consideration of non-truth-functional systems of many-valued logic). The most that can be said is that such a system represents yet another degree of anomaly from the two-valued standpoint, so that its claim to be a system of 'logic' becomes more tenuous. [89] For not only are there orthodox principles of logic (viz., two-valued tautologies) which these systems do not accept – a circumstance virtually inevitable with the many-valued systems in general – but they will have accepted principles (tautologies) that are not tautologies in the two-valued case.

14. THE 'LAW OF CONTRADICTION' IN MANY-VALUED LOGIC

The motivation afforded by the idea of overcoming the classical 'Law of Contradiction' was one major impetus in the development of many-valued logics. There has, however, been considerable disagreement as to the exact content of this law, and it can certainly be construed along various divergent lines. In a canvass of the primary possibilities, it will emerge that the 'Law of Contradiction' can be construed as:

(i) The principle that the thesis

$$p \wedge \neg p$$ [90]

is to be rejected as a logically false (contradictory) proposition.

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[88] *Philosophical Problems of Many-Valued Logic* (*op. cit.*), p. 91.

[89] Inherent in this way of viewing the matter is the idea that the further a system departs from orthodox logic – say by including among its tautologies the outright contradictions of various customary principles say, '$p \wedge \neg p$' or '$\neg (p \veebar \neg p)$' or '$\neg (p \rightarrow p)$' – the more tenuous its claims to constitute a 'logic' will become.

[90] That is, $(\exists p)\ (p \wedge \neg p)$.

(ii) The principle that the thesis

$$\neg(p \wedge \neg p)$$[91]

is to be accepted as an asserted (logically true or tautologous) proposition.

(iii) The principle that one member of the pair p, $\neg p$, must inevitably be false: Not both $/p/ \neq \mathrm{F}$ and $/\neg p/ \neq \mathrm{F}$, that is:

$$(/p/ = \mathrm{F})\text{-or-}(/\neg p/ = \mathrm{F}).$$

(iv) The 'exclusion principle' that p and $\neg p$ cannot both be true together: Not both $/p/ = \mathrm{T}$ and $/\neg p/ = \mathrm{T}$, that is:

$$(/p/ \neq \mathrm{T})\text{-or-}(/\neg p/ \neq \mathrm{T}).$$

(v) It is always false to say that a certain proposition both takes a certain truth-value and yet takes another different one: Whenever $\mathrm{V} \neq \mathrm{V}'$, then one member of the pair

$$/p/ = \mathrm{V}; /p/ = \mathrm{V}'$$

must be false.

The question of whether the 'Law of Contradiction' obtains in many-valued logic is thus a *complex* question that is capable of distinctly divergent interpretations. The following observations are in order:

(1) With respect to construction (i), there is no reason why this version of the principle should not obtain in a system of many-valued logic, and in fact '$p \wedge \neg p$' is a contradiction in most of the systems we have considered. It is not, however, a contradiction in all of them. In particular (and this is of considerable historical importance) it fails for the system $\mathbf{L}_3$ of Lukasiewicz (with F alone antidesignated), as well as for the three-valued systems **B** and **K** of Bochvar and Kleene.

(2) In construction (ii), the principle holds in many systems of many-valued logic. Again, it fails for $\mathbf{L}_3$ (with T alone designated), as well as the three-valued systems **B** and **K**. It should be noted, moreover, that whenever the negation operator $\neg$ is quasi-natural in having the reciprocity feature of taking anti-designated values into designated ones, and conversely, then this second version of the principle is equivalent with the first.

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[91] That is, $(\forall p)\neg(p \wedge \neg p)$.

(3) In construction (iii), the principle comes down to a highly restrictive condition on the truth-table for negation:

p	$\neg p$
T	F
I_1	(1)
I_2	(2)
⋮	
I_n	(n)
F	?

Construction (iii) asserts, in effect that all the positions (1)–(n) must be filled with F's (though we are left free to fill in the ? as we please).

(4) Construction (iv) represents a very weak (and highly plausible) construction of the principle. All of the many-valued systems we have considered obey this version of the principle, and for a very good reason. In this version the principle would be violated only if it happened that the truth-tables for $\neg$ had the feature that:

$$\neg T = T.$$

But in this case $\neg$ could hardly be conceived of as representing as a mode of 'negation'.

(5) It should be noted that:

(iii) yields (iv).

(6) In consideration (v), the principle comes down to legislating the two-valued point of view with respect to one special category of propositions, viz., truth-value assignment statements. Given this construction of the principle, if a statement of the form

$$|p| = V$$

is ever true (and if not, we cannot say that a family of many-valued truth-values is at issue), then all statements of the form

$$|p| = V'$$

will be false whenever $V \neq V'$. In short, only the classical truth-values T and F come into view when the question is one of the truth-value states of truth-value assignment propositions. However, this plausible feature (which holds for virtually all systems of many-valued logic considered to date) can be shown not to be a necessary characteristic of many-valued logics in general.

(7) Constructions (i)–(ii) differ from the rest in that both negation and conjunction is involved. (All the others involve only negation.)

We may thus see that there are several distinct versions of the 'Law of Contradiction' for many-valued logic, and that only one version of the principle – viz., construction (iv) – can plausibly be held to represent a necessary feature of many-valued logics in general.

It remains to reconsider the topic from a variant prespective. Let us assume the many-valued point of view with the classical *truth* and *falsity* generalized to a multiplicity of *designated* (truth-like) and *antidesignated* (false-like) truth-values. Some of the aforementioned versions of the Law of Contradiction can then be reshaped accordingly.

Let us introduce two items of notation, namely let:

$D(S)$ = the designated truth-values of the system S

$D^*(S)$ = the antidesignated truth-values of the system S.

We can now effect a reformulation of two of the versions of the Law of Contradiction into the following counterparts:

(iii′) The principle that one member of the pair p, $\neg p$ must inevitably take on an antidesignated truth-value:

$$[/p/ \in D^*(S)]\text{-or-}[/\neg p/ \in D^*(S)].$$

(iv′) The 'exclusion principle' that p and $\neg p$ cannot both take on designated truth-values:

$$[/p/ \notin D(S)]\text{-or-}[/\neg p/ \notin D(S)].$$

The following observations are in order:

(1) Principle (iii′) can be looked on as a procedural rule governing the designation-status of truth-values: *Be sure to antidesignate the truth-values so that either* V *or* $\neg$V *is antidesignated in every case.* The point is that we must here be

sure (subject to the guidance of the truth-table for negation) that sufficiently many truth-values have been antidesignated.

(2) By contrast, Principle (iv) has the character of an economy principle. This too can be looked on as a procedural rule governing the designation-status of truth-values: *Be sure to designate truth-values sufficiently sparingly that it never happens that* V *and* $\neg$V *are both designated.* The point is that we must here be sure (subject to the guidance of the truth-table for negation) that sufficiently few truth-values have been designated.

(3) Although (iii) yields (iv), it is definitely not the case that (iii′) yields (iv′). On the other hand, (iii′) and (iv′) will be equivalent in the special (quasi-natural) case when D* is taken to be the set-complement of D.

In concluding this section, it is appropriate to remark that in many-valued logic the 'Law of Contradiction' is best looked on as a restrictive principle governing the mode of negation at issue. It seeks to assure that the truth-table for negation is relatively well behaved (in one of several ways) with respect to the distribution of T's and F's – or of designated and antidesignated truth-values.

15. THE 'LAW OF THE EXCLUDED MIDDLE' IN MANY-VALUED LOGIC

The rubric 'Law of the Excluded Middle' actually applies to a variety of distinct though related principles. It can, *inter alia*, be construed as:

(i) The principle that the thesis

$$p \veebar \neg p$$[92]

is to be included among the assertions (i.e., tautologies) of a logical system.

(ii) The 'principle of bivalence' or the '*tertium non datur* principle' to the effect that a proposition must be either true or else false:

$(|p| = \mathrm{T})$-or-$(|p| = \mathrm{F})$.

This 'law of an excluded third' in effect rules out any other truth-values than T and F.

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[92] That is, $(\forall p)\,(p \veebar \neg p)$.

(iii) The 'law of excluded middle' proper, to the effect that of a proposition and its contradictory *at least one* must be true:

$$(/p/ = \mathrm{T})\text{-or-}(/\neg p/ = \mathrm{T}).$$

When this is supplemented by a version of the 'law of contradiction' to the effect that a proposition and its negation cannot both be true, it then follows that the italicized '*at least one*' of the formula can be strengthened to '*exactly one*'.[93]

(iv) A proposition is true if and only if its negation (contradictory) is false, and conversely:

$$/p/ = \mathrm{T} \quad \text{iff} \quad /\neg p/ = \mathrm{F}$$
$$/\neg p/ = \mathrm{T} \quad \text{iff} \quad /p/ = \mathrm{F}.$$

(v) Every proposition either takes a given truth-value or else does not: statements of the type

$$/p/ = \mathrm{V}$$

are invariably either true or false.

The question of whether the 'Law of the Excluded Middle' can be maintained in many-valued logic is thus also a *complex* question that is capable of distinctly divergent interpretations. The following observations are in order:

(1) With respect to construction (i), there is no reason of principle why this version of the principle cannot be maintained in a system of many-valued logic: in fact, '$p \veebar \neg p$' is an asserted thesis (tautology) in various such systems. On the other hand, it must also be recognized that this formula may very well fail to secure the status of an asserted thesis (tautology) in perfectly viable systems of many-valued logic (e.g., $\mathbf{L}_3$).

(2) In construction (ii), the principle cannot be maintained in any system of many-valued logic whatsoever. Indeed, the deliberate violation of this principle is the very basis of and reason for the construction of systems of *many*-valued logic.

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[93] For a historical discussion of (ii) and (iii) in the light of their relationship to the issue of future contingency see N. Rescher, 'An Interpretation of Aristotle's Doctrine of Future Contingency and Excluded Middle' in his *Studies in the History of Arabic Logic* (Pittsburgh, 1963), pp. 43–54.

(3) With respect to construction (iii), there is again no reason why a system of many-valued logic could not incorporate this version of the principle, though, of course, not all will do so, as we have seen. Consider the truth-table for any normal mode of many-valued negation:

p	$\neg p$
T	F
I_1	(1)
I_2	(2)
⋮	⋮
I_n	(n)
F	T

The applicability of the principle at the extreme positions being assured by normalcy, it remains to consider positions (1) – (n). All of these must, according to the principle, be filled by the entry T.

(4) Version (iv) of the principle can readily be maintained in systems of many-valued logic. Consider the truth-table for the negation operator:

p	$\neg p$
T	(1)
I_1	?
I_2	?
⋮	⋮
I_n	?
F	(2)

By the first half of the principle, F must occur at position (1) and *can occur at this position only*. By the second half, T must appear at position (2) and *can occur at this position only*. Hence all the positions marked ? will have to be occupied by 'intermediate' truth-values distinct from the classical T and F. It follows that version (iv) of the principle will obtain in any and every system of many-valued logic whose truth-table for negation is both normal and K-regular (and so always gives intermediate outputs when there are intermediate inputs).

(5) Version (v) of the principle can be maintained with respect

to many of the systems of many-valued logic studied to date. But it is surely not an essential principle which must in the nature of things obtain in all cases.

(6) Version (i) is an object-language formulation of the principle, version (ii)–(iv) are meta-language formulations, and version (v) is a meta-meta-language formulation.

(7) Version (i) involves both negation and disjunction (at the object-language level), and versions (iii) and (iv) involve only negation. Only versions (ii) and (v) involve no special reference to any connectives of the system.

(8) These versions of the principle are all independent except for the following significant interrelationships (and their consequences):

(ii) and (iv) yield (iii)
(iii) and (iv) yield (ii)
(ii) yields (v)

} thus (ii) and (iii) are equivalent given (iv).

We thus see that of the many possible alternative versions of the 'law of the excluded middle' the only one that *must* inevitably be yielded up in the context of a many-valued logic is the 'principle of bivalence' (or *tertium non datur* principle), although the others *may* be given up in certain cases. Abandonment of the Law of the Excluded Middle is therefore far from being an automatic result when we take the step from two-valued to many-valued logic.

Before leaving this topic, one further line of consideration must be taken up. Once a thoroughly many-valued point of view has been assumed, and the ideas of definite *truth* and *falsity* generalized to those of *designated* (truth-like) and *antidesignated* (false-like) truth-values, then some of the foregoing versions of the Law of Excluded Middle can be reshaped accordingly. As above, let:

$D(S)$ = the designated truth-values of the system **S**

$D^*(S)$ = the antidesignated truth-values of the system **S**.

Then reformulate the above into counterparts:

(ii′)

$[/p/ \in D(S)]$-or-$[/p/ \in D^*(S)]$

(iii′)

$[/p/ \in D(S)]$-or-$[/\neg p/ \in D(S)]$

(iv′)

$$|p| \in D(S) \quad \text{iff} \quad |\neg p| \in D^*(S)$$
$$|\neg p| \in D(S) \quad \text{iff} \quad |p| \in D^*(S).$$

(1) Principle (ii′) can be looked on as a procedural rule governing the designation of truth-values: *Be sure to antidesignate all nondesignated truth-values.* As we have seen many-valued logics in some cases do obey this rule, but need not invariably do so. (Note that whenever D* is taken as simply the complement of D, conformity to this rule is assured.)

(2) Principle (iii′) can also be taken as a procedural rule governing the designation of truth-values: *Be sure to designate the truth-values so that either* V *or* $\neg$V *is designated in any given case.* Again, many-valued systems of logic in some cases do obey this rule, but need not invariably do so. For example $\mathbf{L}_3$ does not do so on Lukasiewicz' own approach: Since I is not designated, neither I nor $\neg$I(=I) are designated. (Of course, if one should choose also to designate I, then $\mathbf{L}_3$ will conform to the rule.)

(3) Choices about designation and antidesignation alone can thus assure principles (ii′) and (iii′). This is also true with regard to (iv′), but in a much more restrictive way. Consider, e.g., Post's cyclical negation in a 3-valued case:

p	$\neg p$
1	2
2	3
3	1

Suppose $1 \in D$. Then $2 \in D^*$, since $\neg 1 = 2$. Then $3 \in D$, since $\neg 2 = 3$. But then $1 \in D^*$, since $\neg 3 = 1$. But then $2 \in D$, since $\neg 1 = 2$. But then $3 \in D^*$ since $\neg 2 = 3$. Thus *all* the truth-values will belong to D and all truth-values also belong to D*. But this result is patently unacceptable, since it defeats the whole idea of designation. The lesson is clear. We can *always* assure (iv′) by an appropriate designation policy, viz., to designate and antidesignate each and every truth-value. But a *reasonable* working-out of (iv′) – in those plausible cases in which at least some truth-values are not designated and at least some are not antidesignated – will require a suitably well-behaved mode of negation.

PART III

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Note: Although this bibliography emphasizes the post-1950 period, a number of earlier entries are included to provide a rounded and historically synoptic view of the subject. For a complete bibliography see N. Rescher, *Many-Valued Logic* (New York, 1969). The present chapter draws upon material prepared for this book.

CHAPTER VII

VENN DIAGRAMS FOR PLURATIVE SYLLOGISMS

1. PLURATIVE PROPOSITIONS AND SYLLOGISMS

By a 'plurative syllogism' we understand a two-premiss argument in which, in addition to the familiar categorical propositions of the types *A*, *E*, *I*, and *O*, there may also figure *plurative propositions*[1] of these four types:

U: Most *S* is *P*
W: Most *S* is not *P*
U' (not-*U*): Half-or-more *S* is not *P*
W' (not-*W*): Half-or-more *S* is *P*.

We construe 'Most *X* is *Y*' to assert that *more X's are Y than are Y** (where *Y** is the set-complement of *Y*). Since this can be put in the form 'the cardinality of the set *XY* is greater than that of *XY**', plurative propositions can also be construed for infinite classes. The set *A-E-I-O-U-W-U'-W'* is closed under negation.

An example of a typical plurative syllogism – and in fact a valid one – is the *AUU*-1 syllogism:

All *M* is *P*
Most *S* is *M*

Most *S* is *P*

This example illustrates how the concepts of mood and figure are readily extended to plurative syllogisms.

2. VENN-STYLE DIAGRAMS

The reader will be assumed to be familiar with the well-known device of

[1] We owe this term to correspondence with P. T. Geach. The treatment of such propositions goes back to the Middle Ages at least, and some discussion of the matter can be found, for example, in Averroes' *Quaesita in libros logicae Aristotelis*, in *Aristotelis opera cum Averrois commentariis*, Vol. I (Venice, 1562; photoreprinted, Frankfurt am Main, 1962).

Venn diagrams for testing the validity of classical syllogisms. We shall now develop means for extending this device to plurative syllogisms involving U and W premisses (not – be it noted – U' and W' premisses).[2]

The new item of machinery will be an arrow connecting two line-segments, with that toward which the arrow points to be called the *head* of the arrow, and that from which it points to be called the *vane* of the arrow. This notation is to be superimposed upon the usual machinery of Venn diagrams using circles to represent the extensions of syllogistic terms, with stars and shading to indicate the non-emptiness or emptiness (respectively) of regions. The function of such an arrow is to *indicate that the region comprising all the sectors into which the vane falls is of greater cardinality than the region comprising all the sectors into which the head falls.* Thus the plurative proposition 'Most S is P' is to be represented by Diagrams I and II.

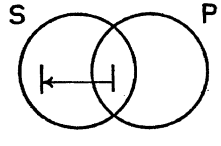

Diagram I

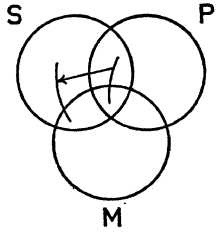

Diagram II

Special Rules for the Arrows

The arrow-notation presented in the preceding section is subject to five rules:

(R1) The vane of an arrow may always be extended.

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[2] We could complicate the machinery introduced here by letting a *plain* arrow from region A to region B mean 'There are at least as many A's as B's' and letting a *flagged* (or barred) arrow mean 'There are more A's than B's'. (A flagged arrow now serves the function of the plain arrow in our text.) Then we can easily diagram U' and W': to 'negate' an arrow we reverse its direction and 'alter' its flagging (i.e., flag it if unflagged, unflag it if flagged). We can now also accommodate Z-propositions of the form 'There are just as many X's as Y's by an (unflagged) arrow pointing in both opposite directions. (However, it seems that no simple diagrammatic procedure is available for accommodating Z', the negate of Z.)

(R2) The head of an arrow may always be contracted.

(R3) The vane of an arrow may always be contracted out of a shaded region.

(R4) The head of an arrow may always be extended into a shaded region.

(R5) An arrow may always be drawn from a starred region into a shaded one.

The appropriateness and validity of these rules should be obvious. (They are not, however, complete in allowing for all valid arrow-operations. For example it would be appropriate to permit both head and vane of an arrow to be extended into any one adjacent region into which neither falls.)

We shall need to make use also of a rule to the following effect:

(R6) In diagramming the premisses of a plurative syllogism, *if* (1) both the heads of the two arrows overlap in one region, and (2) both the vanes overlap in one region, and (3) the head of each arrow overlaps in one region with the vane of the other, *then* the mark of non-emptiness can be placed in the vane-overlap region.

This rule is justified on the basis of R2 and R3 by a *reductio ad absurdum* argument. The situation envisaged in the rule is exemplified by Diagram III:

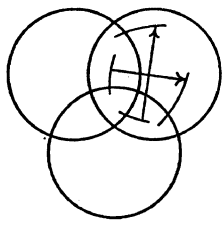

Diagram III

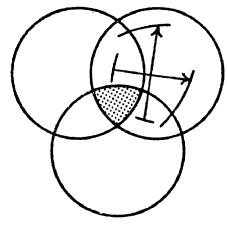

Diagram IV

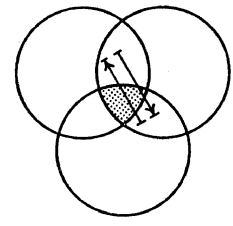

Diagram V

Assume now that the region of vane overlap is empty (shaded in). Then by rule (R3) we obtain Diagram IV. But now by rule (R2) we obtain Diagram V, which is absurd, given the intended meaning of the arrows.

We shall also stipulate a convention: In diagramming plurative syllogisms, the diagramming of the categorical premisses should be done before that of the plurative premisses. This convention is merely an expedient device for avoiding messy diagrams, and is not rendered necessary by any logical considerations.

3. VALIDITY TESTING

Given our arrow-notation and its rules, the process of a Venn-diagrammatic "testing" of plurative syllogisms in which no primed plurative propositions occur is the standard and familiar one of (1) recording in the three-circle diagram the information afforded by the premisses of the syllogism, and then (2) verifying by inspection whether the situation claimed by the conclusion in fact obtains. For example, consider the aforementioned *AUU*-1 syllogism, together with its diagram, Diagram VI:

All M is P
Most S is M

Most S is P

Now since the head of the diagram can be contracted and its vane extended, we obtain Diagram VII, which tells us that the conclusion 'Most S is P' is indeed warranted.

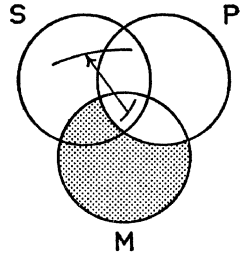

Diagram VI

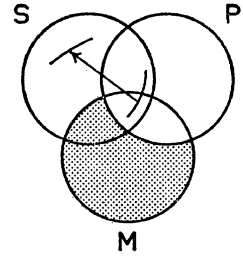

Diagram VII

Reduction of the Primed Cases

Our arrow technique is not designed to accommodate the primed plurative propositions U' and W'. We shall now show that this is a venial shortcoming from the standpoint of the validity-testing of plurative syllogisms.

For it can be established that the following principle obtains:

> Every valid plurative syllogism in which primed propositions occur is either (i) obtained trivially from a valid unprimed plurative by either (a) the strengthening of a premiss (*I* to *W'* or *O* to *U'*) or (b) weakening the conclusion (*A* to *W'* or *E* to *U'*) or (ii) obtained from an unprimed plurative syllogism (or both).

To show this, we remark that plurative syllogisms can be grouped into the following three cases:

Case A: *No primes in the premisses.*

Now if the conclusion is not primed, we are home. If it *is* primed, then the syllogism can be valid only if this primed conclusion can be obtained by weakening a validly drawn unprimed conclusion (case (i-b) above).

Case B: *Exactly one primed premiss.*

B-1: *Unprimed conclusion.*

A universal conclusion is now impossible (given the primed premiss), so the conclusion must be *I*, *O*, *U*, or *W*. Assume it is *I* or *O*; then it must also be obtainable if the primed premiss is weakened (case (i-a) above). So assume the conclusion is *U* or *W*. Then there are four possibilities:

	(1)	(2)	(3)	(4)
Primed Premiss	*U'*	*U'*	*W'*	*W'*
Conclusion	*U*	*W*	*U*	*W*

In case (1) there can be no valid syllogism, for if there were, we could strengthen the premiss *U'* to *E* and weaken the conclusion *U* to *I* and the resulting syllogism would have to be valid. In case (4) there can be no valid syllogism, for if there were, we could strengthen the premiss *W'* to *A* and weaken the conclusion *W* to *O*. In cases (2) and (3) the absence of a valid conclusion can be shown by considering the exactly-half case.

B-2: *Primed conclusion.*

Here antilogism is operative (case (ii) above).

Case C: *Both premisses primed.*

That no valid conclusion can be drawn in this case can be shown by taking both premisses to represent the exactly-half case.

This completes the demonstration of the principle stated at the outset of this section.

4. COMPLETENESS OF THE METHOD

To show the completeness of our diagrammatic testing procedure for unprimed plurative syllogisms (i.e., those involving no primed propositions) two points must be established: (i) Whenever an unprimed plurative syllogism may be claimed to be valid on the basis of diagrammatic test, it is in fact valid. (ii) Whenever an unprimed plurative syllogism is in fact valid, this validity can be exhibited by means of a diagrammatic test. It is not necessary to undertake any further consideration of point (i), which is rendered obvious by the very design of the diagrammatic test. However, point (ii) presents greater difficulty. We shall proceed by cases. Valid unprimed plurative syllogisms may be divided into three groups:

(A) *Trivial* ones, in which either (1) the conclusion of a valid (standard) categorical syllogism is weakened from 'all' to 'most' (*A* to *U*) or from 'no' to 'most-not' (*E* to *W*), or (2) a premiss is strengthened from 'some' to 'most' or from 'some-not' to 'most-not'.

(B) *Quasi-trivial* ones, obtained from a valid (standard) categorical syllogism by a change from 'all' to 'most' or from 'some' to 'most' uniformly in the conclusion and in the major premiss.

(C) *Non-trivial* ones, falling into neither of the above groups, of which (as can be verified by checking the possibilities) there are exactly two, namely *UUI*-3 and *WUO*-3:

Most *M* is *P*	Most *M* is not *P*
Most *M* is *S*	Most *M* is *S*
Some *S* is *P*	Some *S* is not *P*

We shall deal with each of these three cases in turn.

Case A. It is readily verified that Rules R3 and R4 for arrow-manipulation suffice for the diagrammatic validation of syllogisms of this sort: the diagram of the premisses either exhibits or can be transformed to exhibit the diagram for the conclusion.

Case B. Here a case-by-case check can serve to show the adequacy of the method. (We omit the necessary but tedious details.)

Case C. This especially interesting case constitutes the *raison d'être* of Rule R6. The two syllogisms of this case (*UUI*-3 and *WUO*-3) can be validated diagrammatically by this rule.

Although our diagrammatic technique is (demonstrably) complete for

testing the validity of *syllogistic* inferences, it would have to be supplemented to become capable of accommodating certain *asyllogistic* arguments. For example, consider the premisses 'Most X is Y' and 'Most X is Z', shown in Diagram VIII. Since $d+c>a+b$ and $b+c>a+d$ we obtain

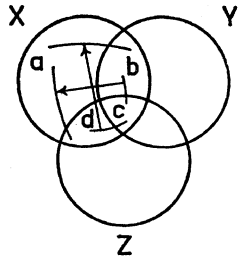

Diagram VIII

$c>a$, i.e., our premisses entail 'There are more XYZ's than XY^*Z^*'s'. But this inference is not comprehended within our diagram-technique as formulated above. (We owe this example to P. T. Geach.)

5. CONCLUSION

It is readily shown that (if the A's are a *proper* subset of the entire domain of discourse, and the cardinality of this domain is *not* specified as some finite number) 'Most A's are B's' cannot be defined by means of the *usual resources of quantificational logic*, or any other type of quantification for that matter (understanding quantifiers to be defined with respect to the entire domain at issue). Nevertheless, the logic of the plurative propositions 'Most S is P' and 'Most S is not P' is an extremely simple matter. For example, as we have shown, syllogisms involving such propositions are subject to a validity test using Venn-diagrams (by our suitably elaborated employment of arrows to indicate the comparative size of two regions of the diagram). More generally, the machinery needed for the analysis of such syllogisms is much less than is required for De Morgan's 'numerically definite syllogisms'.[3]

[3] *Formal Logic* (La Salle, Ill., Open Court, 1926), Chap. VIII. Compare also Henry A. Finch, 'Validity Rules for Proportionally Quantified Syllogisms', *Philosophy of Science* **24** (1957) 1–18.

Consider the two arguments:

All A's are B's	Most C's are A's
All parts of A's are parts of B's	Most C's are B's
	Some A's are B's

Modern textbooks often charge that traditional logic is 'inadequate' because it cannot accommodate patently valid arguments like the first. But this holds equally true of modern quantificational logic itself, which cannot accommodate the second. Powerful tool though it is, quantificational logic is unequal to certain childishly simple valid arguments, which have featured in the logical literature for over a century (i.e., since the days of De Morgan and Boole). Plurative syllogisms afford an interesting instance of an inferential task in which the powerful machinery of quantificational logic fails us, but to which the humble technique of Venn diagrams proves adequate.[4]

[4] This chapter presents details of a finding previously announced by the author in an abstract entitled 'Plurality Quantification', *The Journal of Symbolic Logic* **27** (1962) 372–374. It was originally published as a joint paper with Neil A. Gallagher in *Philosophical Studies* **16** (1965) 49–55. Its authors take pleasure in acknowledging helpful suggestions by Nuel D. Belnap, Jr.

CHAPTER VIII

CAN THERE BE RANDOM INDIVIDUALS?

Mathematicians, logicians, and others occasionally use the idea of 'random individuals' in their expositions, and sometimes even have resort to these for serious theoretical purposes. The object of the present note is to demonstrate the logical impropriety of such procedure.

It will be convenient for my purpose to consider the idea of a 'random' or 'arbitrary' individual in the light of a passage from Professor I. M. Copi's fine textbook, *Introduction to Logic* (New York, Macmillan, 1953). The construction I shall place on this passage is that which, superficially, may seem the most natural to it. It is not, however, the one Copi intends, nor is it requisite, or even proper, for the system of natural deduction he is concerned to present. My purpose in using this passage is thus not to tax Copi with a logical blunder – he is quite innocent of that – but solely to afford myself the expository luxury of a convenient hook on which to hang the argument that I wish to present here.

A geometer, seeking to prove that all triangles possess a certain property, may begin with the words: 'Let *ABC* be any arbitrarily selected triangle.' Then the geometer begins to reason about the triangle *ABC*, and establishes that it has the property in question. From this he concludes that *all* triangles have this property.... We wish now to introduce a notation analogous to the geometer's in talking about 'any arbitrarily selected triangle *ABC*'.... We shall use the (hitherto unused) small letter '*y*' to denote any *arbitrarily selected* individual. We shall use it in a way similar to that in which the geometer used the letters '*ABC*'. Since the truth of any substitution instance of a propositional function follows from its universal quantification, we can infer the substitution instance which results from it by replacing '*x*' by '*y*', where '*y*' denotes *any arbitrarily selected* individual... We may add... [the converse] principle to our list of elementary valid argument forms, stating it as: From the substitution instance of a propositional function with respect to the name of *any arbitrarily selected* individual one can validly infer the universal quantification of that propositional function. This new elementary valid argument form may be written as:

$$\frac{\phi y}{\therefore\ (\forall x)\,\phi x}$$

where '*y*' denotes *any arbitrarily selected individual.* [*Op. cit.*, pp. 293–295.]

Let us accept this discussion in a naive and literal way, and adopt the usage of special symbols that denote any 'arbitrarily selected' individual.

At once trouble is upon us. For what can an 'arbitrarily selected' individual be like? Consider a set $S=\{1, 2, 7, 8, 13\}$. Let 's' denote an 'arbitrarily selected element' of S. Is s prime? Is $s<9$? Is s even? Is s odd? Is $s=7$? All of these questions, it is clear, must be answered negatively – otherwise s ceases to be 'randomly' or 'arbitrarily selected'. Thus down this road of 'arbitrarily selected' individuals there lies only confusion and paradox. Somehow s must be an element of S. But it is a queer, shadowy sort of element indeed. For S has only 5 elements, yet s – though an element! – is not identical with any one of them. Here, surely, is a perfect paradox. For s is to be an element of S. But it is not 1, not 2, not 7, etc. Yet 1, 2, 7, etc. are the *only* elements of S!

The reader will at once recognize this line of argument as familiar. It is, in fact, merely a crib of Berkeley's attack upon abstract ideas, upon 'the general idea of a triangle – which is neither oblique nor rectangular, equilateral, equicrural nor scalene, but all and none of these at once' (Introduction to the *Principles*). Berkeley rejects as absurd an *abstract idea* of a particular thing, such as a triangle, because it demands an impossible coexistence of incompatible properties. And just this is the case also with 'arbitrary' or 'random' individuals.

The foregoing case against the idea of a 'random' or 'arbitrary' element of a set can be recast, with all due logical rigor, against 'random' or 'arbitrary' *individuals*. Let me return to the idea of an 'arbitrarily selected individual' to be denoted by a special symbol, say 'y'.

In any satisfactory system of quantificational logic we must inevitably have the inference of 'universal instantiation',

$$\text{(I)} \qquad \frac{(\forall x)\,\phi x,}{\therefore\ \phi z},$$

i.e., if *everything* has a certain property, then any (particular) individual has this property. Let us now also assume the mode of argument characteristic of the 'arbitrarily selected individual' y,

$$\text{(II)} \qquad \frac{\phi y}{\therefore\ (\forall x)\,\phi x},$$

i.e., if our 'random' or 'arbitrarily selected' individual has a certain property, then every individual must have this property.

From (II) we at once obtain, by *modus tollens*, the derived mode of argument,

$$\text{(III)} \qquad \frac{(\exists x) \sim \phi x}{\therefore \sim \phi y},$$

i.e., if some individual lacks a certain property, then the 'arbitrarily chosen' individual y cannot have this property.

In any system of quantification logic adequate to universes of discourse that contain more than one thing, the following will be an accepted, or at least *acceptable* assertion:

$$\text{(1)} \qquad (\forall x)(\exists z) \sim (x = z).$$

From (1) we at once obtain, by (I):

$$\text{(2)} \qquad (\exists z) \sim (y = z).$$

But now, by (III), (2) entails:

$$\text{(3)} \qquad \sim (y = y).$$

This paradoxical-seeming result leads to outright contradiction. For the logic of quantifiers cannot evade

$$\text{(4)} \qquad (\forall x)(x = x),$$

and this, by (I), yields

$$\text{(5)} \qquad y = y.$$

But (5) contradicts (3). Thus the concept of a 'randomly' or 'arbitrarily selected' *individual* swiftly leads to outright contradiction (at any rate in universes with more than one thing).

The lesson of this line of argument is that any talk of 'randomly' or 'arbitrarily selected individuals' is thoroughly inept. When 's' is used to denote an 'arbitrary element' of $S = \{s_1, s_2, s_3, \ldots\}$ what is intended is simply that a statement about s is to be true about every element of S: s is not an element of S – even though '$s \in S$' is true! – because a statement involving 's' is a shorthand synopsis of a multiplicity of statements, *viz.* the corresponding assertions regarding each particular S-element. Such

an s is not a *thing* or 'element' or 'individual' at all, it is a notational device: s is a universal surrogate for individuals, and not itself an individual. This is how it comes about that '$s=s_1$' and '$s\neq s_1$' are *both* false. There is not here some sort of infraction of the law of contradiction; for the former statement, without its misleading notation, amounts to '$(\forall x)(x=s_1)$' and the latter to '$(\forall x)(x\neq s_1)$'. And note that, once we cease to view y as an *individual*, rule (III) becomes wholly improper, for it licenses the inference from '$(\exists x)\sim\phi x$' to what is *now*, in effect, '$(\forall x)\sim\phi x$'.

As we have said, to speak of a 'random individual' is not to speak of an individual at all. When one introduces random individuals, one can do so meaningfully only subject to the self-denying ordinance represented by the convention that: *Nothing is to be said about a random individual that is not intended about ALL of the individuals of the domain at issue.* A random individual is therefore not a thing but a linguistic principle, a shorthand device for presenting universal statements.

To regard a 'random element' as an *element* or a 'random individual' as an *individual* is to commit what Whitehead terms the 'fallacy of misplaced concreteness' and involves what philosophers have come to call a *category mistake*. A statement like 'ϕy' does *not* say something about a peculiar 'random individual' y: it says that the property ϕ characterizes every particular element of our universe of discourse. There are no 'random' or 'arbitrarily selected' individuals, just individuals. The 'arbitrariness' or 'randomness' resides not in individuals, but in the deliberate ambiguity of the notation by which reference to them is made. To talk of 'random' or 'arbitrarily selected' *individuals* is to reify a notational device. And this, in the present instance, is not merely unwarranted, it is demonstrably absurd.[1]

[1] This chapter is a slightly amplified version of a paper of the same title in *Analysis* **18** (1958) 114–117.

CHAPTER IX

THE LOGIC OF EXISTENCE

1. INTRODUCTION

In recent philosophical discussion regarding the nature of the concept of existence and the logical theory of cognate concepts, the tendency has been to view these problems in their Kantian setting, by focussing upon the question 'Is "exists" a predicate?' and the bearing which these considerations have upon the validity of the ontological argument.[1] Now it is, to be sure, inevitable that the logical theory of *existence* must in the last analysis rest upon *ontological* and not purely logical considerations. As Russell has said, 'Pure logic has no occasion for names, since its propositions contain only variables. However, the logician may wonder in his unprofessional moments, what constants could be substituted for his variables.'[2] But it is clearly anomalous to hinge the entire theory of existence primarily or largely upon *only one* of its ontological implications. And this is especially inadmissable when it happens to be what is, after all, the most remote, abstruse, and metaphysical of the problems here involved. Furthermore, a denial that 'exists' is a predicate, if stemming solely from repugnance to the ontological argument and desire for its overthrow, would be highly captious, and would be as wholly indefensible as any other purely *ad hoc* determination made solely for the purpose of evading an unwanted consequence. It is necessary to find other reasons, wholly independent of the ontological argument, upon which a denial that 'exists' is a predicate can reasonably be based. However crucial the question whether 'exists' is or is not a predicate may be for the validity of the ontological proof, there is surely no justifiable course but to base this decision, in the final analysis, upon other grounds. Pivoting the logical

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1 See, for example, the following: H. S. Leonard, 'The Logic of Existence', *Philosophical Studies* 7 (1956) 49–64 (especially pp. 57–58); G. Nakhnikian and W. C. Salmon, '"Exists" as a Predicate', *The Philosophical Review* **66** (1957) 535–542.

2 *Human Knowledge* (New York, 1948), p. 74.

theory of existence about the ontological argument is to put the cart before the horse; they go together, it is true, but not in *that* order.

The purpose of the present chapter is to suggest a more judicious mode of treatment. I propose here to examine the logic of existence concepts, proceeding insofar as possible in total abstraction from the ontological argument. Intrinsic merits aside, this approach will, I think, prove fruitful in terms of deepened insight into fundamental aspects of the logical theory of existence and of denotation.

The notation 'E!a' will here serve to represent the statement 'a exists'. This notation is not intended to prejudice – and surely cannot appropriately be interpreted as prejudicing – the question 'Is "exists" a predicate?'[3] Clearly, the proposed abbreviation is wholly incapable of bearing upon this question, which after all does not relate to such purely superficial considerations of notation or typography, but deals with the fundamental logical and conceptual issues that are here at stake. No one who has denied that existence is a predicate is, or to my knowledge has been, of a mind to pivot this thesis about the question of whether existential statements can be abbreviated.

The notation 'E!a' exemplifies (but does not exhaust) the possibility of attributions – in this case of 'existence' – that are not necessarily predications. The justification of this distinction is the wish to keep open the possibility of a theory of predication that calls for predicates to have characteristics not possessed by attributes in general, that is, by all attributes. There is no *a priori* reason for constraining ourselves to having to say about every attribution whatsoever those things which we want to say about predications. For this is not a matter which can be settled adequately and defensibly in advance of a study of the logic of predication. It is thus potentially useful, and for our purposes necessary, to adopt some such terminology as that of employing the (generic) term *attribute* in application to the broad genus of attributions in general, and to reserve the term *predicate* for the species of property-denoting terms.

This distinction between the genus *attribute* and its species *predicate* must at once be carried over into the theory of abstraction. If A is some attribute or other, we must not now regard its intensional abstract $(\lambda x)(Ax)$

[3] Sometimes this question is somewhat naively taken as settled affirmatively by the possibility of such a notation. Both of the papers cited in note 1 appear to provide instances of this.

as a property, but rather as again merely an attribute. Abstraction (namely, intensional abstraction) does not automatically yield properties; only the intensional abstract of a *predicate* is a property.

Thus a second-order functional calculus that can serve as a fruitful framework of reference for a discussion of whether 'exists' is or is not a predicate will have to provide two kinds of first-order functional variables, those which stand generally for 'mere' attributes (i.e., the genus, including the species predicate), and the predicate-variables proper. Here the Greek capitals 'Φ', 'Ψ',..., will be used as attribute variables, and the Greek lower-case letters 'ϕ', 'ψ',..., will serve as predicate variables. The fact that every predicate is an attribute can now be represented as:

(1) $\quad (\forall\phi)\,(\exists\Phi)\,(\phi = \Phi)$.

The *denial* of the proposition that every attribute is a property appears as

(2) $\quad \sim(\forall\Phi)\,(\exists\phi)\,(\phi = \Phi)$,

or, equivalently,

(3) $\quad (\exists\Phi)\,(\forall\phi)\,(\phi \neq \Phi)$.

Again, it will now be a general truth for any sentential function '...' of one individual variable (say x) that:

(4) $\quad (\exists\Phi)\,(\Phi = (\lambda x)\ldots)$.

However, the counterpart of (4) with a predicate variable 'ϕ' in place of the attribute variable 'Φ', that is,

(5) $\quad (\exists\phi)\,(\phi = (\lambda x)\ldots)$,

will not be true in general, but only under some condition on the sentential function '...' which assures that its intensional abstract is a predicate.[4]

What criterion is to be used in discriminating between attributes which are not predicates and those which are? Until this question has been

[4] The dualism of sets and classes in the set-theory of von Neumann is analogous with this predicate versus attribute distinction: every set is a class, but only under special conditions are classes sets. J. von Neumann, 'Über eine Widerspruchsfreiheitsfrage in der axiomatischen Mengenlehre', *Journal für reine und angewandte Mathematik* **160** (1929) 227–241. Improved formulations of this system are due to P. Bernays and K. Gödel. P. Bernays, 'A System of Axiomatic Set Theory', *The Journal of Symbolic Logic* **2** (1937) 65–77; K. Gödel, *The Consistency of the Continuum Hypothesis* (Princeton, 1940).

answered, the problem of the status of existence as a predicate cannot meaningfully be raised, let alone solved. For this purpose the following, very obvious, negative (and therefore perhaps incomplete) criterion may well prove sufficient:

The attribute Φ_1 will *not* be a property, and so its name will *not* be a predicate, if there exists (in the logical system under discussion) a sentential expression of the form '$\ldots\Phi_1\ldots$' such that either: (1) '$\ldots\Phi_1\ldots$' does *not* entail (logically imply) '$(\exists\phi)(\ldots\phi\ldots)$', or (2) '$(\forall\phi)(\ldots\phi\ldots)$' does *not* entail '$\ldots\Phi_1\ldots$'.

The appropriateness of this criterion is at once obvious. For the existence of an expression of the sort mentioned would show that the attribute Φ_1 does not fall within the range of the predicate-variables of the system, that is, that Φ_1 is not a predicate.[5] Only some such criterion can serve as foundation for a fruitful discussion of whether the *attribute* E! is or is not a *predicate.*

2. THE IDEA OF NONEXISTENT POSSIBLES

We must now turn to the idea of nonexistent possibles. At the outset, we note that the principal consideration that militates on behalf of nonexistent possibles lies at the basis of the concept of modality as such. If it is possible that the world might differ from what it is, it is possible (say) that Queen Elizabeth I of England might not have died childless. But this conception of possible (nonactual) *states of affairs* described in terms of propositional modalities immediately brings with it the conception of possible (unactual) *things* – in our example, the (possible) firstborn of Elizabeth I. It is a familiar, commonplace idea that certain things are possible, though not in fact actual or extant. Winged horses or unicorns, for example, might exist but certainly do not.

An immediate consequence of this idea is that one must reject – in an unqualified form, at any rate – the thesis that *all* things exist. Or rather,

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[5] In making the property-versus-attribute distinction dependent upon the concept of deducibility, this criterion relativizes the distinction to the deductive strength of the logical system under consideration. This makes for an element of conventionalism that may at first view appear undesirable. Against this critique it must be urged that this consequence is not unacceptable because while certain 'conventional' consequences do ensue upon the choice of a particular system, this choice itself is not arbitrary, being conditioned by our informal (unsystematic or, better, presystematic) understanding of the purposes the chosen system is intended to serve.

and more pointedly, one must exclude its formal counterpart '$(\forall x)\ \mathrm{E!}x$' from among the asserted propositions of a logical system that is to be viewed as capable of providing an appropriate formal framework for the analysis of existence-concepts. There are surely true statements which say that certain possible states of affairs might obtain, or certain things might possibly be so, though these are not in fact realized or actual. Indeed, to maintain the actuality of all alternative possibilities is logically untenable in view of the fact that these will mutually exclude one another.

The conception of nonexistent possibles restrains us from accepting the thesis that all things exist: '$(\forall x)\ \mathrm{E!}x$'. Two fundamental factors necessitate this rejection, the one stemming from modal logic, the other deriving from the theory of counterfactual statements. These considerations – both drawn from different branches of logical theory that are seemingly removed from the problem area at issue – must be considered in detail.

It cannot reasonably be gainsaid that there are true statements to the effect that certain things are possible, though not in fact actual or extant. An example of a statement of this kind is the statement that, while unicorns do not exist, it is perfectly possible that they might. There are surely true statements which say that certain possible states of affairs might have obtained, or things might possibly have been, though these are not in fact realized or actual. Indeed to maintain the actuality of all alternative possibilities is logically untenable in view of the fact that these will mutually exclude one another. For reasons such as these one wants to affirm the proposition:

(1) $\quad (\exists x)\,(\Diamond\, \mathrm{E!}x \,\&\, \sim \mathrm{E!}x)$

which in turn entails

(2) $\quad (\exists x) \sim \mathrm{E!}x.$

But (2) clearly contradicts

(3) $\quad (\forall x)\ \mathrm{E!}x.$

We are thus brought to a choice between (1) and (3). Now there is – to my mind – simply no gainsaying the fact that an adequate logic of modality requires acceptance of (1); and there is thus little recourse but to reject (3).

My second objection to acceptance of '$(\forall x)\ \mathrm{E!}x$' is based upon the consideration that there are true but counterfactual existential statements.

The following is surely an instance of this kind: 'If Hamlet had actually existed, he could not have been a more complex personality than the protagonist of Shakespeare's play'. Now the analysis of counterfactual statements is a complex and difficult matter into whose technical intricacies I have no desire to enter now.[6] But fortunately the sole feature of these statements required for my present purpose is virtually the only point on which general agreement among all the diverse accounts obtains. If a counterfactual statement of the form 'If P were to have been realized, then Q would also have been realized' is true, this requires that the statement P involved in the protasis or antecedent of the counterfactual statement must be false. But consider now any counterfactual statement with an existential protasis, such as that given above. Such a statement S has an antecedent of the form '$\mathrm{E!}x_1$'. By the very nature of counterfactuals, we therefore at once have:

(4) $\qquad S \supset \sim \mathrm{E!}x_1.$

Now this entails:

(5) $\qquad \mathrm{E!}x_1 \supset \sim S.$

Since, in general, we have

(6) $\qquad (\forall x)\ \mathrm{E!}x \supset \mathrm{E!}x_1,$ [7]

it follows at once that:

(7) $\qquad (\forall x)\ \mathrm{E!}x \supset \sim S.$

Thus the assertion of '$(\forall x)\ \mathrm{E!}x$' precludes *ab initio* the truth of any counterfactual existential statement whatsoever. Since this consequence cannot be regarded as acceptable, we have here yet another line of argument in support of the rejection of: $(\forall x)\ \mathrm{E!}x$.

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[6] The pioneer studies are: R. Chisholm, 'The Contrary-to-Fact Conditional', *Mind* **55** (1946) 289–307; and N. Goodman, 'The Problem of Counterfactual Conditionals', *The Journal of Philosophy* **44** (1947) 113–128. A particular helpful survey of the extended literature is: E. F. Schneider, 'Recent Discussions of Subjunctive Conditionals', *The Review of Metaphysics* **6** (1952) 623–647. See also N. Rescher, *Hypothetical Reasoning* (Amsterdam, 1964).

[7] This presupposes a system of logic such that nonexistent individuals – such as Bucephalus or Hamlet – lie within the range of the individual variables 'x', 'y', etc.

3. DEFINITIONS OF E!

The rejection of '$(\forall x)\mathrm{E}!x$' entails rejection of a variety of proposed definitions of 'E!' which have been discussed in the recent literature. For example, it has been proposed that 'E!' be characterized by the definition:

(E1) E! for '$(\lambda x)\,(x = x)$'. [8]

But in view of the inevitability of the assertion '$(\forall x)\,(x = x)$' in any system of quantificational logic, (E1) has as its immediate consequence that $(\forall x)\mathrm{E}!x$. (E1) is therefore open to the line of objection developed in Section 2.

Again, a *prima facie* plausible definition of 'E!' is the following:

(E2) E! for '$(\lambda x)\,(\exists y)\,(y = x)$'.

But here again the definition founders on the fact that in the standard systems of quantification logic (for non-empty domains) it is a theorem that $(\forall x)\,(\exists y)\,(y = x)$. And thus (E2) also has the unwanted consequence $(\forall x)\,\mathrm{E}!x$. (Note our abandonment – for the rest of the present chapter – of a rigoristic use of quotes around theses.)

The rejection of (E2) does *not*, it should be noted, of itself amount to a rejection of Quine's well-known thesis, 'To be is to be the value of a variable'.[9] For if this thesis is interpreted conservatively as stating

(Q) $\mathrm{E}!x \rightarrow (\exists y)\,(y = x)$,

then it does not entail $(\forall x)\mathrm{E}!x$. However, the *converse* of Q, namely,

(Q′) $(\exists y)\,(y = x) \rightarrow \mathrm{E}!x$,

which Quine apparently also intends, *does*, as we saw in the foregoing argument concerning (E2), lead to $(\forall x)\,\mathrm{E}!x$, and must be rejected on that ground. (The reader to whom the rejection of (Q′) appears as a paradox must abate his impatience until pp. 156–157, where a rationale for this rejection is developed.)

I have myself elsewhere proposed to define 'E!' by the definition:

(E3) E! for '$(\lambda x)\,[(\exists \phi)\,(\sim \phi x \,\&\, \Diamond\,(\exists y)\,\phi y)]$'. [10]

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[8] G. Nakhnikian and W.C. Salmon, '"Exists" as a Predicate', *The Philosophical Review* **66** (1957) 535–542 (see p. 539).

[9] W. V. Quine, 'Designation and Existence', *The Journal of Philosophy* **36** (1939) 701–709 (see especially pp. 707–708). This article is reprinted in H. Feigl and W. Sellars, eds., *Readings in Philosophical Analysis* (New York, 1949).

[10] 'Definition of "Existence"', *Philosophical Studies* **7** (1957) 65–69 (see pp. 67–69).

According to this definition, the individual x 'exists' if there is some property which fails to characterize x, but which *could* be instantiated. (Note that (E3) could be formulated as $(\lambda x)\,[(\exists \phi)\,(\phi x \,\&\, \Diamond\,(\exists y) \sim \phi y)]$, i.e., as stating that x exists if it has a *non-trivial* property, one which need not characterize all individuals whatsoever.) Let us examine the proposition $\sim(\forall x)\,\mathrm{E}!x$, with 'E!' taken in the sense of (E3). This is equivalent with:

(1) $\quad (\exists x)\,(\forall \phi)\,[\Diamond\,(\exists y)\,\phi y \supset \phi x]$.

Now we cannot, in any adequate theory of modality, avoid assent to the proposition: $(\exists \phi)\,[\Diamond\,(\exists y)\,\phi y \,\&\, \Diamond\,(\exists y) \sim \phi y]$. But this, together with (1), leads to

(2) $\quad (\exists x)\,(\exists \phi)\,(\phi x \,\&\, \sim \phi x)$,

which is absurd (i.e., self-contradictory). Thus (E3) leads to the consequence $(\forall x)\,\mathrm{E}!x$, a result which renders this definition itself unacceptable.

A definition of 'E!' recently proposed by H. S. Leonard is, essentially, as follows:

(E4) $\quad$ E! for '$(\lambda x)\,[(\exists \phi)\,(\phi x \,\&\, \Diamond \sim \phi x)]$'. [11]

On this definition, x 'exists' only if it has a *contingent* property, that is, a property which it has, but not necessarily. Consider now the statement $(\exists x) \sim \mathrm{E}!x$, with 'E!' taken in the sense of (E4), that is:

(3) $\quad (\exists x) \sim (\exists \phi)\,(\phi x \,\&\, \Diamond \sim \phi x)$.

This is equivalent with:

(4) $\quad (\exists x)\,(\forall \phi)\,(\phi x \supset \Box\, \phi x)$.

Now (4) amounts to the assertion that there are objects which necessarily have each of the properties they do in fact have. This is wholly acceptable, and indeed is quite true of abstract mathematical objects.[12] Thus (E4) passes muster from the standpoint of avoiding the consequence: $(\forall x)\,\mathrm{E}!x$.

But while (E4) appears as acceptable from this standpoint, it has one unfortunate consequence which necessitates its rejection also.[13] For it is

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[11] 'The Logic of Existence', *Philosophical Studies* 7 (1957) 49–64 (see especially p. 58).

[12] Indeed just this is the thesis upon which rests the objection to (E4) shortly to be developed.

[13] This objection to (E4) was raised in my paper on 'Definitions of Existence', *Philosophical Studies* 7 (1957) 65–68 (see p. 67). This motivated my proposed (and here rejected) definition (E3).

at least an arguable thesis that abstract mathematical objects, such as numbers and sets, exist. But such objects necessarily have each of those properties which they do have. All of the properties of the number 3, for example, being half of 6, are necessary (demonstrable).[14] For such an abstract object X we have:

(5) $\quad (\forall \phi)\,(\phi X \supset \Box \phi X)\,.$

This is equivalent to

(6) $\quad \sim (\exists \phi)\,(\phi X \,\&\, \Diamond \sim \phi X)\,,$

which is equivalent to '$\sim \mathrm{E!}X$', when 'E!' is taken in the sense of (E4). Thus (E4) in effect denies *a priori* and without further ado or qualification that abstract mathematical objects exist. This, I submit, warrants its rejection.

To make a fresh start upon this vexing question of the definition of 'E!', let us consider first of all the thesis that if a thing does not exist, then its only *qualitative* properties[15] are those which characterize all objects. A symbolic transcription of this thesis is:

(T) $\quad (\forall x) \sim \mathrm{E!}x \supset (\forall P)\,[Px \supset (\forall y)\,Py]\,.$

The limitation of (T) to qualitative properties derives from the consideration that its general counterpart

(T.1) $\quad (\forall x) \sim \mathrm{E!}x \supset (\forall \phi)\,[\phi x \supset (\forall y)\,\phi y]\,,$

or equivalently

(T.2) $\quad (\forall x) \sim \mathrm{E!}x \supset (\forall \phi)\,[(\exists y) \sim \phi y \supset \sim \phi x]$

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[14] This of course is not true of the *symbol* (numeral) used to represent this number, which has various contingent properties, such as occurring (or not occurring) upon this page. Such a contingent property of the representing symbol does not induce a contingent property of the abstract object, i.e., it is not a contingent property of the number 3 that it is represented by a symbol occurring on this page. This last proposition is not, however, a truth of symbolical theory, but a complex philosophical thesis, viz., Platonism. For my argument above it is not requisite to maintain that Platonism is true, but simply that it is tenable or arguable, i.e., not to be rejected on solely logical grounds.

[15] *A qualitative property* (for which I shall here use variables 'P', 'Q', etc.) is a property denoted by a predicate which either (1) is a primitive predicate of the language, or (2) is definable in terms of primitive predicates by means of alternation and conjunction (only), in terms of these alone, and thus without negation and without any reference to particular individuals.

leads to difficulties. For if a property ϕ_1 is in the range of our predicate-variables 'ϕ', 'ψ', etc., so must $\sim\phi_1$ be. Thus if we grant, as we must, that there is a property ϕ_1 such that both $(\exists y)\sim\phi_1 y$ and $(\exists y)\,\phi_1 y$, then it is clear that the right-hand side of (T.2) is false on logical grounds alone, so that (T.2) leads to the unacceptable $(\forall x)\,\mathrm{E!}x$.[16]

The thesis (T) has the consequence:

(1) $\quad (\forall x)\,[(\exists P)\,[Px\,\&\,(\exists y)\sim Py]\supset \mathrm{E!}x]$.

This at once suggests the possibility of defining 'E!' as follows:

(E) $\quad$ E! for '$(\lambda x)\,(\exists P)\,[Px\,\&\,(\exists y)\sim Py]$'.

This definition – which specifies existents as objects that have nonuniversal qualitative properties – differs from the definition (E3) of the preceding section in referring to different properties and in deleting the '$\Diamond$' of that definition. Thus the definition modifies somewhat the circumstances of E!'s applicability – a thing we saw to be necessary.

Let us check the adequacy of (E). First, does it entail $(\forall x)\,\mathrm{E!}x$? To examine this, let us consider the proposition $(\exists x)\sim\mathrm{E!}x$, with 'E!' taken in the sense of (E), namely:

(2) $\quad (\exists x)\sim(\exists P)\,[Px\,\&\,(\exists y)\sim Py]$.

This is equivalent to:

(3) $\quad (\exists x)\,(\forall P)\,[Px\supset(\forall y)\,Py]$.

Now (3) would be self-contradictory only if its denial, that is,

(4) $\quad (\forall x)\,(\exists P)\,[Px\,\&\,(\exists y)\sim Py]$,

were to be a logical truth; that is, if every individual were, on solely logical grounds, to have a *qualitative* property not possessed by all others, just as

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[16] Another reason for confining this thesis to qualitative properties is the following: suppose that (*T*) applied, for example, to $(\lambda x)\,(x=x_1)$, where $\sim\mathrm{E!}x_1$; i.e., to the property of being some particular nonexistent object. Then (*T*) would lead to: $\sim\mathrm{E!}x\supset(x\neq x_1)$. We would thus be forced to a choice between (1) denying the self-identity of nonexistents, or (2) accepting $(\forall x)\mathrm{E!}x$. In this context, it is of interest to note that if nonexistence were to be regarded as a qualitative property, i.e., if $\sim$ E! were to fall within the range of the variables '*P*', '*Q*', etc., then (*T*) would entail: $\sim\mathrm{E!}x\supset[\sim\mathrm{E!}x\supset(\forall y)\sim\mathrm{E!}y]$. This consequence is not acceptable, for, since we cannot evade denial of $(\forall y)\sim\mathrm{E!}y$, this leads to $(\forall x)\mathrm{E!}x$.

it logically must uniquely have the (non-qualitative) property $(\lambda x)\,(x = x_1)$, where x_1 is the individual in question. But this cannot reasonably be held to be a logical truth, and so (E) does not entail $(\forall x)\,\mathrm{E!}x$.

Again, let us see if – as with (E4) of the preceding section – (E) denies existence to abstract objects which (such as numbers) have all of their properties by necessity. Is it possible for an individual object x_1 to satisfy

$$(5) \qquad (\forall \phi)\,(\phi x_1 \supset \Box\, \phi x_1),$$

when also

$$(6) \qquad (\exists P)\,(Px_1 \,\&\, (\exists y) \sim Py)?$$

This would be impossible only if (5) and (6) were incompatible with one another, that is, if their conjunction were self-contradictory, which is not the case. For if (5) entailed not-(6), then we would have the implication:

$$\text{If } (\forall \phi)\,(\phi x_1 \supset \Box \phi x_1) \text{ then } (\forall P)\,(Px_1 \supset (\forall y)\,Py).$$

But this entailment would hold only if '$\Box Px_1$' entailed '$(\forall y)\,Py$', that is, if the only necessary qualitative properties an individual object could have are those that are universal. Thus to maintain the incompatibility of (5) and (6) would be to deny the possibility of differentially necessary qualitative properties – those essential to (i.e., necessary for) certain objects but not others. But this is a controversial ontological thesis that can surely not be advocated on logical grounds alone. (We shall return to this topic in Sect. 7 of Chapter X below.)

Thus acceptance of (E) as a definition of 'E!' appears to be entirely consonant with all of the foregoing considerations as to the logic of the existence-concept. For, in particular, it entails neither $(\forall x)\mathrm{E!}x$, nor does it necessitate a denial of existence to abstract objects.

4. THE THEORY OF DESCRIPTIONS

I propose next to consider the bearing of the foregoing considerations regarding the logic of existence upon the theory of descriptions. I will begin by accepting the position, widely held among logicians, that an adequate theory of description or designation must be such that the meaningfulness or the logical appropriateness of designating expressions ought not to

hinge upon questions of existence.[17] One must be able to discuss, designate, or describe objects without in any way prejudicing the question of their existence. The propriety of an object description ought not to be dependent upon questions of existence. Designation must not prejudge existence: we want to be able to refer to 'the object having a certain property' without thereby either affirming or denying (tacitly or otherwise) that this object *exists*. In short, we do not want a theory of designation that has the consequence:

(1) $\quad (\forall\phi)\,\mathrm{E!}\,[(\imath x)\,\phi x]\,.$

In place of (1), a theory of designation that is to be satisfactory from the standpoint of existence considerations ought to accept only some weaker counterpart which qualifies ϕ by some additional existential requirement. We want therefore in place of (1) a substitute of the conditional form:

(2) $\quad (\forall\phi)\,(\ldots \supset \mathrm{E!}\,[(\imath x)\,\phi x])\,.$

The determination of the condition '...' must of course depend upon the definition of the description operator '$\imath$'.

Consider the contextual definition of '$\imath$' given in *Principia Mathematica*:[18]

(D) $\quad [(\imath x)\,\phi x]\,F[(\imath x)\,\phi x]$ for '$(\exists y)\,(\forall x)\,[(\phi x \equiv (x = y))\,\&\,Fy]$',[19]

where F is understood to be an arbitrary propositional function of one variable. By this definition we have:

(3) $\quad \mathrm{E!}\,[(\imath x)\,\phi x] \equiv (\exists y)\,(\forall x)\,[(\phi x \equiv [x = y])\,\&\,\mathrm{E!}\,y]\,.$

From (3) we derive

(4) $\quad (\forall\phi)\,\{(\exists y)\,(\forall x)\,[(\phi x \equiv [x = y])\,\&\,\mathrm{E!}y] \supset \mathrm{E!}\,[(\imath x)\,\phi x]\}\,,$

which has the form of (2). The antecedent of (4) thus gives the condition

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[17] Among those who take this view are Frege, Russell, and Quine; Hilbert and others, however, dissent.

[18] See *14.01.

[19] The scope notation on the left-hand side of this definition is necessary to provide a clear indication of the propositional function F in which substitution is intended. Otherwise the notation, e.g., $\sim F[(\imath x)\varphi x]$, would be equivocal between (1) $\sim(\exists y)\,(\forall x)\,[(\varphi x \equiv (x = y))\,\&\,Fy]$ on the one hand, and (2) $(\exists y)\,(\forall x)\,[(\varphi x \equiv (x = y))\,\&\sim Fy]$ on the other. See Chapter III of the Introduction to the second edition of *Principia Mathematica* (Cambridge, 1925).

which, *relative to* (D), it is necessary that a predicate ϕ satisfy before we are entitled to assert that E! $[(\imath x)\ \phi x]$. With a different definition of '$\imath$', the form of this condition would, of course, also be changed. But so long as this condition is not trivial (i.e., universally satisfied), the theory of designation in question does not lead to (1), and is thus unobjectionable from the standpoint of the existential considerations here in view.

It is of interest to explore the possibility of a theory of designation which avoids existential commitments altogether, even in the sense, weaker than that of the preceding section, of avoiding the consequence:

(1) $\quad (\forall\phi)\ (\exists y)\ (y = (\imath x)\ \phi x)$.

Now any contextual definition of '$\imath$' which, like that of *Principia Mathematica*, has the general form,

(2) $\quad F\,[(\imath x)\ \phi x]$ for '$(\exists y)\ [\ldots\ \&\,Fy]$',

will, letting F be $(\lambda z)\ (z = (\imath x)\ \phi x)$, have the consequence,

(3) $\quad [(\imath x)\ \phi x = (\imath x)\ \phi x] \equiv (\exists y)\ [\ldots\ \&\, y = (\imath x)\ \phi x]$.

Since the left-hand side of (3) is an inevitable tautology, (4) entails

(4) $\quad (\forall\phi)\ (\exists y)\ (y = (\imath x)\ \phi x)$,

so that there is here no possibility of rejecting (1).

This line of reasoning suggests the possibility of a contextual definition of '$\imath$' which, instead of having the form (2), is of the form:

(5) $\quad F[(\imath x)\ \phi x]$ for '$(\forall y)\ [\ldots \supset Fy]$'.

Now a simple definition of this kind is:

(D′) $\quad [(\imath x)\ \phi x]\ F[(\imath x)\ \phi x]$ for '$(\forall y)\ (y \in \hat{x}\phi x \supset Fy)$'. [20]

This definition entails the following specification of $(\imath x)\ \phi x$: (i) when there is exactly one individual, say a, such that ϕa, then $(\imath x)\ \phi x = a$;[21]

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[20] Here again the scope notation is necessary for reasons analogous to those explained above $\hat{x}(\ldots x\ldots) = \{x \mid \ldots x\ldots\}$ = the set of all x such that $\ldots x\ldots$. This definition was initially proposed by me in the paper, already cited, on 'Definitions of "Existence"'. I was led to it not by the present line of thought, but by the considerations discussed in connection with propositions (6) and (7) below.

[21] In this case $\hat{x}\varphi x = \{a\}$. Now let F be $(\lambda z)\ [(\imath x)\ \varphi x = z]$. Then (D′) yields

(5.1) $\quad [(\imath x)\ \varphi x = (\imath x)\ \varphi x] \equiv (\forall y)\ [y \in \{a\} \supset (\imath x)\ \varphi x = y]$,

which entails $(\imath x)\ \varphi x = a$.

(ii) when there is no individual for which ϕ obtains, $F[(\imath x)\,\phi x]$ is always true, for any propositional function F,[22] and finally (iii) when there are several individuals – say the entire set $S=\{a, b, \ldots\}$ – such that ϕx obtains, then $F[(\imath x)\,\phi x]$ is true or false according as F does or does not obtain for all the elements of S.[23]

This definition of '$\imath$', like that of *Principia Mathematica*, has the characteristic contextualistic feature of Russell's classic theory of descriptions,[24] namely, while it always determines a truth-status for statements of the form '$F[(\imath x)\,\phi x]$', it does not invariably specify $(\imath x)\phi x$ as some existing thing. In this regard (D′), as well as the *Principia* definition, differs from those of Frege and of Quine. On Frege's definition,[25] $(\imath x)\,\phi x$ is a, Λ, and the class S, respectively in the three cases, and on Quine's definition,[26] $(\imath x)\,\phi x$ is a, Λ, and Λ, for these three cases.

The choice between various definitions of definite description is, of course, in a sense a matter of 'convention', since no one definition can be proved to be 'the correct' one. But of course the choice between alternative definitions is not purely arbitrary. We have distinct, although informal, guiding conceptions regarding the consequences to which an adequate theory of descriptions should lead, and evaluate alternate definitions in terms of the informal, and as it were pre-theoretic, acceptability

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[22] In this case $\hat{x}\varphi x = \Lambda$. Here (D′) becomes

(5.2) $\quad F[(\imath x)\,\varphi x] \equiv (\forall y)\,[y \in \Lambda \supset Fy]$,

whose right-hand side is trivially true, since '$y \in \Lambda$' is uniformly false.

[23] In this case (D′) becomes

(5.3) $\quad F[(\imath x)\,\varphi x] \equiv (\forall y)\,(y \in S \supset Fy)$,

whose right-hand side is true if, and only if, F holds for all elements of S. This concept of description is closely analogous with the concept of *restricted quantification* due to Theodore Hailperin (*The Journal of Symbolic Logic* **22** (1957) 19–35). Hailperin introduces the 'restricted quantifier', '$(vx\varphi x)$' in such a way that '$(vx\varphi x)\,Fx$' amounts to 'for every x such that φ (if any), Fx'. Thus '$F[(\imath x)\,\varphi x]$', with description taken in the sense of (D′), amounts to Hailperin's '$(vx\varphi x)\,Fx$'. The motivation that led Hailperin to the concept of restricted quantification is essentially identical with that leading to (D′), viz., the avoidance of existential presuppositions. The writer wishes to take this opportunity to thank Professor Hailperin for his very helpful comments on an early draft of this paper.

[24] This is set forth in his paper 'On Denoting', *Mind*, N.S. **14** (1905) 479–493.

[25] *Grundgesetze der Arithmetik* (Jena, 1873), Vol. 1, p. 19.

[26] *Mathematical Logic* (Cambridge, 1947), pp. 147–149. A convenient comparative discussion of alternative definitions of '$\imath$' is §§ 7–8 of R. Carnap's *Meaning and Necessity* (Chicago, 1947).

of the theories to which they give rise in their consequences. And this acceptability may in turn depend upon the theoretical purposes in view. It may, for example, differ as between contexts of mathematical elegance and philosophical congruity.

The definition (D′), it should here be noted, has one advantage *vis à vis* its alternatives. It guarantees generally and without further ado that if every object having a certain property ϕ has also another property ψ, that is, if $(\forall x)(\phi x \supset \psi x)$, then the object specified as $(\imath x)\,\phi x$ also must have the property ψ; in other words, it guarantees that:

(6) $\qquad (\forall x)(\phi x \supset \psi x) \supset \psi[(\imath x)\,\phi x]$.

To see that this is so, we must first note that, in general,

(7) $\qquad (\forall x)(\phi x \supset \psi x) \supset (\forall y)[y \in \hat{x}\phi x \supset \psi y]$,

and that (D′) thus gives us:

(8) $\qquad \psi[(\imath x)\,\phi x] \equiv (\forall y)(y \in \hat{x}\phi x \supset \psi y)$.

But (7) and (8), taken together, entail (6).

This fact yields to (D′) a certain advantage of analogy, for (6) clearly holds for the nondegenerate cases of descriptive designation, and a definition of '$\imath$' which extends this into a general principle gains added plausibility (other things being equal). It is a significant advantage of our (D′) over Russell's (D) in point of naturalness that such statements as the following will all be true:

The present king of France is a king.
The present king of France is male.
The present king of France is French.

But of course the consequence that *all* affirmative predications about nonexistents are true is unnatural – just as it is an unnatural feature of Russell's theory of descriptions that they are all false. It should be noted also that (6) entails

(9) $\qquad (\forall\phi)\,\phi[(\imath x)\,\phi x]$,

which again is a 'natural' and thus desirable feature of the definition (D′).

I wish now to draw attention to some consequences of the theory of description based on (D′) which are of particular interest for the logical theory of existence. I will here talk about properties which are *not* ex-

emplified, such as 'being a unicorn', so that there exist no objects that have them. Here then we have $\hat{x}\phi x = \Lambda$. In such cases, as we have seen, the theory of descriptions based on (D′) leads to the consequence that for *any* propositional function F, '$[(\imath x)\ \phi x]\ F[(\imath x)\ \phi x]$' is *always* true, that is, that any statement 'about' $(\imath x)\ \phi x$ (in this sense) is true. We are here preserved from outright contradiction by the scope notation, because for any unrealized property ϕ, both '$(\forall y)\ (y \in \hat{x}\phi x \supset Fy)$' and '$(\forall y)\ (y \in x\phi x \supset \sim Fy)$' are true, due to the falsehood of their antecedents. The contradictory of '$[(\imath x)\ \phi x]\ F[(\imath x)\ \phi x]$' is *not* '$[(\imath x)\ \phi x] \sim F[(\imath x)\ \phi x]$', but is '$\sim [(\imath x)\ \phi x]\ F[(\imath x)\ \phi x]$'.

Thus if description is defined by means of the definition here in view, we will have, for example, that for unrealized ϕ,

$$(10) \qquad (\imath x)\ \phi x \in \Lambda,$$

however odd-seeming, is true and perfectly innocuous. And, of course, (10) does not lead to $(\exists x)\ x \in \Lambda$.

Basing our discussion on the concept of description set forth in the preceding section, let us resume the topic of nonentities.

5. DESCRIPTION AND NONENTITIES

There is no reason whatsoever why we should not regard such items $(\imath x)\ \phi x$, where ϕ is unrealized, as *nonexisting things*, that is, as purely conceptual objects.

Among modern logicians, Quine especially has been concerned to argue against such a conception of possible (but nonexistent) objects. Thus in his very amusing and well-written paper 'On What There Is', he writes,

> Possibility, along with the other modalities of necessity and impossibility and contingency, raises problems.... But we can at least limit modalities to whole statements. We may impose the adverb 'possibly' upon a statement as a whole, and we may well worry about the semantical analysis of such usage; but little real advance in such analysis is to be hoped for in expanding our universe to include so-called *possible entities*. I suspect that the main motive for this expansion is simply the old notion that Pegasus, e.g., must be because it would otherwise be nonsense to say that he is not.[27]

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[27] *Review of Metaphysics* 2 (1948) 21–38; see p. 22. Reprinted in *Semantics and the Philosophy of Language*, ed. by L. Linsky (Urbana, 1952), pp. 189–206.

However this line of argument will not do at all, because the doctrine of nonexistent possibles can without difficulty be formulated in sentences with suitable use of modalities. The following statement represents the essential condition under which an object n is a nonexistent possible:

(N) $\quad \sim (\exists x)\,(x = n) \,\&\, \Diamond\,(\exists x)\,(x = n)\,.$

This asserts that there exists (in the sense of existential quantification) nothing identical with n, although it is possible that there might. Now we must be careful to refrain from the temptation to infer, from (N), that:

(N′) $\quad (\exists y)\,[\sim (\exists x)\,(x = y) \,\&\, \Diamond\,(\exists x)\,(x = y)]\,.$

For (N′) entails '$(\exists y)\,(\forall x)\,(x \neq y)$' which is incompatible with the thesis '$(\forall x)\,(x = x)$'. But this merely goes to show that nonexistent possibles cannot be taken to lie within the range of values of our individual variables, 'x', 'y', and so forth, so that Existential Generalization is not valid with respect to such objects as the n of (N). The self-contradictory character of (N′) must *not* be construed to show that there 'are' no nonexistent possibles, but only that they are not in the range of values of individual variables.

We must therefore recognize that, if ϕ is not exemplified, the 'conceptual object' $(\imath x)\,\phi x$ does not fall within the range of our individual variables 'x', 'y', 'z', ... Thus Existential Generalization is simply not valid with respect to $(\imath x)\,\phi x$, so that – as remarked above – (1) does not entail:

(2) $\quad (\exists x)\, x \in \Lambda.$

Again, when ϕ is not realized by any object, '$\mathrm{E!}\,[(\imath x)\,\phi x]$' is true. Thus '$\mathrm{E!}\,[(\imath x)\,\phi x]$' entails neither '$(\exists x)\,\phi x$' nor '$(\exists x)\,\mathrm{E!}\,x$'. And, more generally, '$\psi\,[(\imath x)\,\phi x]$' entails neither '$(\exists x)\,\phi x$' nor '$(\exists x)\,\psi x$'. We have quite succeeded in our goal of deliberately keeping our theory of descriptions free from existential presuppositions and implications.

In his paper 'On What There Is', Quine has great fun with possible (nonexistent) entities.

> Take, for instance, the possible fat man in that doorway; and, again, the possible bald man in that doorway. Are they the same possible man, or two possible men? How do we decide? How many possible men are there in that doorway? Are there more possible thin ones than fat ones? How many of them are alike? Or would their being alike make them one? Are no *two* possible things alike? Is this the same as saying that it is impossible for two things to be alike?

Or, finally, is the concept of identity simply inapplicable to unactualized possibles? But what sense can be found in talking of entities which cannot meaningfully be said to be identical with themselves and distinct from one another.[28]

But once we have made an end to the amusement to be had in this way, we must face the sobering thought that the denial of possible entities, that is, the thesis that all things exist, also leads to thoroughly unpalatable consequences, and in a much less frivolous way (as I have tried to show above).

After all, the problems so amusingly posed by Quine pose no insuperable theoretical difficulties. How many possible objects are there? As many as can be described. When are two possible objects alike? When their descriptions entail a similarity. When are two possible objects identical? When their defining descriptions are logically identical (equivalent). The doctrine of possible objects poses no profound theoretical difficulties, for everything save existence alone (and its implications) remains precisely as with objects which 'really' exist.

* * *

In a logical system in which an *explicit* treatment of the conceptual objects of the type $(\imath x)\ \phi x$ (where $\hat{x}\phi x = \Lambda$) is to be possible, we are called on to introduce a new set of variables, 'η', 'ζ', 'ξ',..., with the convention that, while

(1) $\quad (\forall x)\,(\exists y)\,(x = y)$

is true, so is

(2) $\quad (\exists \eta)\,(\forall x)\,(\eta \neq x)$.

Nor is there any reason why we should not have

(3) $\quad (\exists \eta)\,(\exists \zeta)\,(\eta \neq \zeta)$,

that is, there is no reason why our criterion of identity for these new 'objects' should not be such that $(\phi \neq \psi) \rightarrow [(\imath x)\ \phi x \neq (\imath x)\ \psi x]$, even in the face of $\hat{x}\phi x = \hat{x}\psi x = \Lambda$.[29] These new Greek letter variables, then, would

[28] *Ibid.*, 23–24; L. Linsky, *op. cit.*, pp. 189–206.

[29] This leads to the strong criterion of identity that descriptions are identical only when the describing properties are (intensionally) identical. Clearly only some such strong intensional mode of identity can serve as a basis for discriminating among nonexistents.

have for their range not only the existing objects, but also a class of (various) objects which, for want of a better word, might (in view of (2) above) be termed nonentities.

The nonentities, it can be shown, are all 'elements' of the null class. For consider once again the definition (D') in the case of properties ϕ for which $\hat{x}\phi x = \Lambda$. Letting Fz be $z \in \Lambda$, we here obtain

(4) $\quad [(\imath x)\ \phi x \in \Lambda] \equiv (\forall y)\ [y \in \Lambda \supset y \in \Lambda],$

which, the right-hand side being patently tautologous, entails:

(5) $\quad (\imath x)\ \phi x \in \Lambda.$

We can thus say, without self-contradiction, that – however great the verbal paradox may seem – the null class, that is, the class containing no (existing) elements, contains all nonentities.

That the seeming paradox of this approach is unreal, that is, that no outright inconsistency of any kind is engendered, can be readily seen. For what we have in effect done is to go over to a universe of discourse in which we effect a dichotomy into two mutually exclusive and exhaustive sub-universes, say A (the 'existents') and B (the 'nonentities'). Over the individuals of the first sub-universe we use Roman variables, over those of the union of the two sub-universes, Greek. We have, in effect, simply gone over to a two-sorted logic,[30] using the greater freedom of articulation here afforded to enlarge the meaning-scope of the existence concept represented by existential quantification.

I have elsewhere shown that a system of many-sorted (say n-sorted) logic can be transformed into a system of one-sorted logic, with variables ranging over a single all-inclusive domain.[31] This can be done by means of the introduction of various kinds of quantifiers, each kind pertaining to one of the n domains (i.e., there are n distinct general and correspondingly n distinct existential quantifiers), so that the domain-reference function comes to be absorbed by the quantifiers. With this approach, we can reformulate (1) and (2) above as

(6) $\quad (\forall_1 x)\ (\exists_1 y)\ (x = y),$

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[30] See H. Wang, 'The Logic of Many-sorted Theories', *The Journal of Symbolic Logic* **17** (1952) 105–116.

[31] For details see N. Rescher, 'Many-sorted Quantification', *Proceedings of the 12th International Congress of Philosophy* (Padua, 1958).

and

(7) $(\exists_2 x)(\forall_1 y)(x \neq y)$,

respectively. Here (7), or equivalently,

(8) $\sim(\forall_2 x)(\exists_1 y)(x = y)$,

does *not* contradict (6) because a different type of general quantification is involved.

In such a system, in which the first mode of quantification is reserved exclusively for the domain of existents, we could now define 'E!' by means of the definition:

(9) E! for '$(\lambda x)(\exists_1 y)(y = x)$'.

This does indeed lead to

(10) $(\forall_1 x)\,\mathrm{E!}\,x$,

but this – unlike $(\forall_2 x)\,\mathrm{E!}x$ – is now perfectly acceptable.[32]

It was remarked above that the rejection of

(11) $(\exists y)(y = x) \rightarrow \mathrm{E!}\,x$

has a seeming air of paradox about it. But this can now be smoothed out. For the paradox disappears once we recognize that an equivocation in the meaning of existential quantification is an inevitable result of speaking of two orders of entities: those which (actually) *exist*, and those which only '*subsist*' (i.e., do not exist save as *entia imaginationis*). When this equivocation is removed by explicit introduction of two kinds of existential quantification, then we see that (11) is indeed legitimate and acceptable in the form,

(12) $(\exists_1 y)(y = x) \rightarrow \mathrm{E!}\,x$,

but is not acceptable, and must be rejected in the form,

(13) $(\exists_2 y)(y = x) \rightarrow \mathrm{E!}\,x$.

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[32] Since '$\forall_1$' is intended by stipulation to have reference solely to existents, $(\forall_1 x)(\Diamond \mathrm{E!}x \supset \mathrm{E!}x)$ is now quite innocuous.

6. MACCOLL'S THEORY

The theory of designation and existence which has been articulated in the preceding sections is a modernized recasting of a theory propounded by Hugh MacColl in a series of papers in *Mind* around the turn of the century.[33] To establish this identity I quote a few of the key passages of MacColl's discussion:

> [Let us take our 'Symbolic Universe' or 'Universe of Discourse' to be] composed of all things real or unreal that are named or expressed by words or other symbols in our argument or investigation. By this definition we assume our Symbolic Universe (or 'Universe of Discourse') to consist of our universe of realities r_1, r_2, r_3[34], etc., together with our universe of unrealities [nonentities] u_1, u_2, u_3, etc.... [p. 74.] When they [the majority of symbolic logicians] define Λ (or any other symbol) as indicating non-existence, and then assert that the equivalence $\Lambda = \Lambda \cap A$ is always true, whatever the class A may be, they appear to me to make an assertion which cannot easily be reconciled with their data or definition. For suppose Λ to consist of the three unrealities u_1, u_2, u_3, and the class A to consist of u_3, r_1, r_2, r_3 (one unreality and three realities), the class $\Lambda \cap A$ common to both contains but one individual, the unreality u_3. We cannot therefore say that the class Λ, which contains three individuals, is the same as the class $\Lambda \cap A$, which contains but one. [p. 78.] Can we consistently assert, as the formula $\Lambda \subset A$ asserts, that the unreal (and therefore nonexistent) individuals u_1, u_2, etc., are contained in the class of real individuals r_1, r_2, etc.? It is hardly an answer to say that the symbol Λ, as logicians usually define it, does not denote an unreal class made up of unreal members, as I define it, but a null or empty class containing no members; for is not a null class containing no members equivalent to an unreal class made up of unreal members?... The crucial point which here separates me, I believe, from all other symbolists is that I regard the class Λ, whether empty or made up of unrealities, as necessarily *excluded from every real class*; whereas they regard it as *contained in every class* whether real or not. [p. 578.]

I believe that these quotations suffice for my present aim of showing that the theory of designation and existence discussed above is substantially the same as that propounded by MacColl.

MacColl had no sooner enunciated his theory than it was sharply

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[33] *Mind*, N.S. **14** (1905) 74–81, 295–296, 390–397, 401–402, and 578–580.

[34] In the interests of simplicity and readability I have here, and in a few instances below, modified the notation used by MacColl.

criticized.[35] The principal criticism was that of Bertrand Russell, who rejected it on the ground that it 'conflicts with the law of contradiction':

> Mr. MacColl... regards individuals as of two sorts, real and unreal; hence he defines the null-class as the class consisting of all unreal individuals. This assumes that such phrases as 'the present King of France', which do not denote a real individual, do, nevertheless, denote an individual, but an unreal one. This is essentially Meinong's theory, which we have seen reason to reject because it conflicts with the law of contradiction.[36]

Now the criticism of Meinong to which Russell here refers is this:

> [Meinong's] theory regards any grammatically correct denoting phrase as standing for an *object*. Thus 'the present King of France', 'the round square', etc., are supposed to be genuine objects. It is admitted that such objects do not subsist, but nevertheless they are supposed to be objects. This is in itself a difficult view; but the chief objection is that such objects, admittedly, are apt to infringe the law of contradiction. It is contended, for example, that the existent present King of France exists, and also does not exist; that the round square is round, and also not round; etc. But this is intolerable; and if any theory can be found to avoid this result, it is surely to be preferred.[37]

This criticism of Russell's amounts to rejecting a theory of descriptions based on a contextual definition of '$\imath$' which, like (D′) on page 150, has the consequence that if $\hat{x}\phi x = \Lambda$, then '$F[(\imath x)\ \phi x]$' is always true. This is the basis, and indeed the sole ground, of Russell's objection. But in the first place it is simply false that, verbal oddities aside, such a theory leads to contradiction. And in the second place Russell's objection is open to the decisive counter *argumentum ad hominem* that his *own* theory of descriptions (based on the definition of '$\imath$' given as (D) on page 149) leads to the consequence that in this case '$F[(\imath x)\ \phi x]$' is always false, and this is clearly every bit as much (or little) in conflict with the 'principle of contradiction'. The fact is that in 1905 Russell had not yet recognized the importance of scope notations for the theory of descriptions. His own subsequent discoveries in this connection suffice fully to rehabilitate MacColl's theory of descriptions against his criticisms.

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[35] B. Russell, 'The Existential Import of Propositions', *Mind*, N.S. **14** (1905) 398–401; 'On Denoting', *ibid.*, 479–493 (see especially p. 491); A. T. Shearman, 'Note on Logical Existence', *ibid.*, p. 440.

[36] 'On Denoting', *Mind*, N.S. **14** (1905) 491.

[37] *Ibid.*, pp. 482–483.

In short, neither Russell's nor for that matter any other attack upon MacColl's theory of description and existence can be regarded as successful.

MacColl's insistence that logic must make room not only for existing things, but also for things which are possible (but not actual), is characteristic of his turn of mind. He alone among turn-of-the-century logicians is keenly alive to modal distinctions. Thus he writes:

> They [i.e., other symbolic logicians] divide their propositions into two classes, and *two only*, the *true* and the *false*. I divide propositions not only into true and false, but into various other classes according to the necessities of the problem treated; as for example into *certain*, *impossible*, *variable* [i.e., contingent].[38]

7. CONCLUSION

I wish now, at long last, to bring the discussion back to the ontological argument. The pivotal traditional objection to the ontological proof derives from a philosophical repugnance to the procedure of 'defining into existence'; to affirming that an object exists simply and solely in virtue of the manner in which it is defined, without *ex machina* assumption of any existential presuppositions or postulates. The opponent of the ontological proof rejects the legitimacy of such a procedure, denying that there is any property which, when inserted into the definition of an object as one of its essential attributes, inevitably has the consequence that said object exists. What is denied, then, is that there is a property ϕ such that:

(1) $\quad \Box \mathrm{E}![(\imath x)\, \phi x]$.

We have seen in the foregoing discussion of the theory of description that it is, or can be, a general principle of the logic of definite description that:

(2) $\quad \Box (\forall \phi)\, \phi [(\imath x)\, \phi x]$.

Now if existence is a predicate, that is, if 'E' is held to lie within the range of the predicate variables 'ϕ', 'ψ', and so forth, then (2) leads at once to:

(3) $\quad \Box \mathrm{E}![(\imath x)\, \mathrm{E}!x]$.

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[38] 'Symbolic Reasoning, V', *Mind* **12** (1903) 355–364 (see p. 356).

Therefore, if 'E!' is a predicate, then it can be argued that an object into whose *conception* the property it represents, namely existence, is inserted as an essential, defining property, must necessarily exist.[39]

The denial of the predicational status of existence affords – as Kant rightly saw – an adequate logical strategem for the frustration of the ontological argument. But as regards just this point, our own discussion has been wholly inconclusive. Within the scope of the present investigation we have found no reason either for or against holding existence to be a predicate.

* * *

In conclusion, I should like to review some of the major points that have emerged in the course of the foregoing analysis of existence concepts:

(1) An adequate theory of existence must reject $(\forall x)$ E!x, that is, the thesis that all things exist. It cannot otherwise give due recognition to the fact that there are nonexistent though not unrealizable things and states of affairs.

(2) A satisfactory formal analysis of the concept of existence can be provided in terms of the thesis that nonexistent objects can have no non-universal qualitative properties (i.e., no such properties not possessed by everything).

(3) A theory of description can be built up which (i) involves no existential presuppositions, and (ii) has the feature – despite (2) above – that every statement 'about' (in a suitable sense) nonexistent objects will be true.

(4) A theory of distinct nonexistent objects is possible, but obviously requires a highly intensional criterion of identity for these objects, since extensionally all unrealized descriptions are clearly the same.

(5) The theory of descriptions envisaged in the previous remarks is closely similar in its essentials to a short-lived theory propounded by Hugh MacColl in *Mind* around the turn of the century, and defended by him with poor success against various criticisms by other symbolic logicians, notably Bertrand Russell.[40]

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[39] It should, however, be recalled that, on the theory of descriptions here presented, E! [(ιx) φx] does not entail either $(\exists x)$ φx or $(\exists x)$ E!x. (See Section 4 above.)

[40] This chapter is a somewhat expanded version of a paper 'On the Logic of Existence and Denotation' published in *The Philosophical Review* **68** (1959) 157–180.

CHAPTER X

NONSTANDARD QUANTIFICATIONAL LOGIC

1. QUANTIFICATION OVER ITEMS OTHER THAN 'ENTITIES'

On the standard – and, as we suppose, familiar – approach to quantificational logic one starts with a domain of 'individuals' D, with respect to which one is to construe quantificational statements as follows:

$(\forall x)\ \phi x$ *for* *For every individual* x in the domain D, it is the case that ϕx.

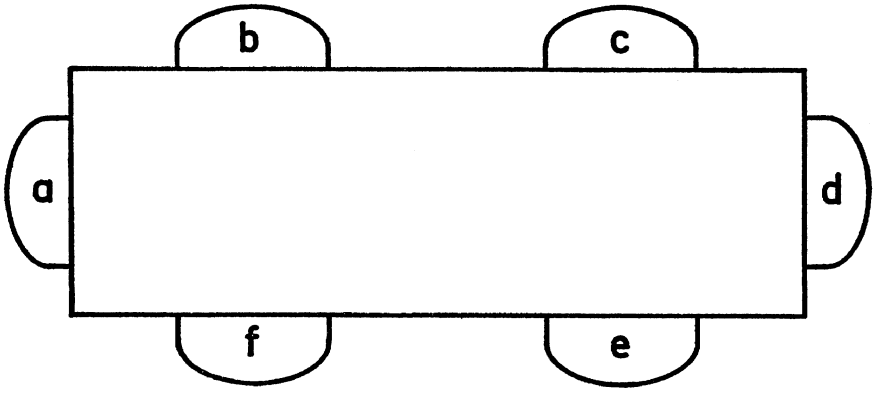
$(\exists x)\ \phi x$ *for* *For some individual* x in the domain D, it is the case that ϕx.

On such an approach, *Quine's dictum* to the effect that 'To be is to be the value of a variable' makes eminently good sense: the things that 'exist' within the purview of the discussion are just exactly the elements of the domain D. But the case becomes a difficult one when the constituents of the domain D are not 'things' in any plausible sense of this very flexible word, when, for example, they are *moments of time*, or *positions in an ordering*, or *numerical indices*, or *propositions*. We shall deal with time in a later chapter, and with indices and propositions in later sections of this present chapter. For the moment, let us confine ourselves to positions in an ordering.

Consider, for example, the domain D consisting of six positions (a)–(f) in a hypothetical table seating arrangement (see page 163). Let Nxy mean 'x neighbors (be it to the right or the left) upon y'. The truth of some propositions regarding N inhere in the very 'logic of the concept': for example the symmetry and irreflexivity of N:

$$(\forall x)(\forall y)\ [Nxy \rightarrow Nyx]$$
$$(\forall x) \sim (Nxx).$$

These truths would inhere generically in the very nature of the concept of neighboring, and require no specific reference to our particular table-arrangement scheme. (Note that they are 'strictly universal truths' – with



all quantifiers universal and placed in front.) But consider by way of contrast, such truths as:

Some position does not neighbor on position (*a*)

$$(\exists x) \sim Nxa.$$

Every position is such that *some* position does not neighbor on it

$$(\forall x)(\exists y) \sim Nxy.$$

Every position has two distinct positions neighboring on it

$$(\forall x)(\exists y)(\exists z)[y \neq z \,\&\, Nxy \,\&\, Nxz].$$

Such nonuniversal statements are clearly contingent upon the specific features of our hypothetical table seating arrangement. Now it is plain as a pikestaff that neither the use of an existential quantifier nor the use of variables ranging over a domain of values here commits us to the 'existence' of any 'entities'. Quantification and the use of variables must therefore not be thought of as involving any essential reference to 'things' or 'entities' that possess any sort of 'ontological status' worth conjuring with. There is no sound basis for denying, on grounds of logical considerations, a highly pluralistic ontology – including not only substance (i.e., individuals proper), but space (i.e., positions), time (i.e., a continuum of moments), propositions, properties (and relations), numbers, etc.

2. PROPOSITIONAL QUANTIFIERS

Perhaps the easiest way in which the step from propositional to quantifica-

tional logic can be taken is to introduce $\forall$ and $\exists$ as respectively universal and existential quantifiers *over the domain D of all propositions*. Whenever we have a tautology (of standard propositional logic), say '$(p \,\&\, q) \supset p$', for example, we would assert the corresponding universal closure:

$$(\forall p)\,(\forall q)\,[(p \,\&\, q) \supset p]\,.$$

For any *contradiction*, say '$(p \,\&\, \sim p) \,\&\, q$', we could assert the *negation* of the corresponding existential closure:

$$\sim(\exists p)\,(\exists q)\,[(p \,\&\, \sim p) \,\&\, q]\,.$$

For a contingent statement, on the other hand, such as '$p \,\&\, q$', we would expect to assert the corresponding existential closure itself:

$$(\exists p)\,(\exists q)\,(p \,\&\, q)\,.$$

With this simple procedure, we could recast all of (standard) propositional logic in quantificational form. All of the other assertable statements of the resulting system QP (for 'quantified propositional logic') would then be derivable from assertions of this basic sort by orthodox quantificational inference. For example, the following is a proof of '$\forall p(\exists q)\,(q \supset p)$':

(1)	$(\forall p)\,(p \supset p)$	since '$p \supset p$' is tautologous
(2)	$p_0 \supset p_0$	from (1) by universal instantiation
(3)	$(\exists q)\,(q \supset p_0)$	from (2) by existential generalization
(4)	$(\forall p)\,(\exists q)\,(q \supset p)$	from (3) by universal generalization.

3. INDEX-QUANTIFICATION

Little attention has been paid by logicians to the device of quantifying – not over *individuals*, but – over *indices*, a device which has found wide employment in mathematics. Consider, for example, an indexed list of the individual (proper) names of certain items:

$$a_1, a_2, a_3, \ldots .$$

Suppose we want to symbolize the statement 'a_1 and a_2 and a_3 all have the property ϕ'. The simplest course is, of course, represented by the conjunction:

$$\phi a_1 \,\&\, \phi a_2 \,\&\, \phi a_3\,.$$

But this conjunctive procedure will not, if the list of a's having ϕ is itself infinite, enable one to say, for example, that all the a's other than a_1 and a_2 have ϕ. Here we have the recourse of constituting a set A to include all the a_i,

$$A = \{a_1, a_2, a_3, \ldots\}$$

and then using quantification in the standard way to express the fact at issue:

$$(\forall x)\,[(x \in A \,\&\, x \neq a_1 \,\&\, x \neq a_2) \supset \phi x]\,.$$

However, we could also bypass quantification over individuals in favor of quantification over subscripts, as follows:

$$(\forall i)\,[(i \neq 1 \,\&\, i \neq 2) \supset \phi a_i]\,.$$

(Here, of course, the indexical variable i must be understood to range over the set of positive integers: $i = 1, 2, 3, \ldots$).

We shall now show how it is possible to employ this subscript-indexing or labeling technique to do the work of predicate logic without using predicates – and, as we have just seen, without quantifying over any domain of individuals.

Let predicates be conceived of as being *listed in extension*, so that instead of the predicate P we have the (labeled) list, (p_1)---, (p_2)---, (p_3)---, ..., that registers all of the individuals to which the predicate at issue applies. This extensional treatment of predicates presupposes, of course, that the predicate has (at least one) application: otherwise we have no list of instances. Thus we dispense with P in favor of a listing of the form (p_i)---, where i belongs to some index-set which need not of course be finite or even denumerable (e.g., we could employ real-number subscripts). The universal or intensionally empty predicate (which applies to everything) corresponds to the entire (labeled) list (u_1)---, (u_2)---, ... We shall adopt the notation that, whenever a predicate P with its corresponding list (p_1)---, (p_2)---, ... is given, '$[p_j]$' is to denote the occupant of the jth place in the p_i-listing, i.e., the item corresponding to the index (p_j). Further we adopt the rule *that every predicate list is of the same length as the list of the* u_i by the artificial device that the last *new* entry in the list is repeated *ad infinitum scilicet ad finem*. Thus if the universal (empty)

predicate corresponds to the (hypothetically all-inclusive) list,

$$(u_1)\, a, \quad (u_2)\, b, \quad (u_3)\, c, \quad (u_4)\, d, \quad (u_5)\, e,$$

then, if the predicate P corresponded in fact to the list,

$$(p_1)\, a, \quad (p_2)\, c,$$

we would, under this convention, represent P as:

$$(p_1)\, a, \quad (p_2)\, c, \quad (p_3)\, c, \quad (p_4)\, c, \quad (p_5)\, c.$$

That is, since $[p_2] = c$ is the last *new* entry, we set $[p_i] = [p_2] = c$ for all $i > 2$.

The only pieces of (extra-propositional) logical machinery to be introduced are quantification over subscripts,[1] and the relation of individual identity.

Any statement formulable within orthodox predicate logic can now be transposed into this framework. For example '$(\exists x)\, Fx$' becomes '$(\exists_i)(\exists_j)([u_i] = [f_j])$,' and '$(\forall x)\, Fx$' becomes '$(\forall i)(\exists_j)([u_i] = [f_j])$'. (To preserve a distinction between necessary and contingent propositions, on such an approach, one must, however, adopt – as is in principle possible – a distinction between necessary and contingent identities.[2]) The standard categorical propositions can be handled as follows:

(A)	All S is P:	$(\forall i)(\exists_j)\ ([s_i] = [p_j])$,
(E)	No S is P:	$(\forall i) \sim (\exists_j)\ ([s_i] = [p_j])$,
(I)	Some S is P:	$(\exists i)(\exists_j)\ ([s_i] = [p_j])$,
(O)	Some S is not P:	$(\exists i) \sim (\exists_j)\ ([s_i] = [p_j])$.

It is an interesting characteristic of this scheme that all of the usual relationships embodied within the traditional 'square of opposition' will obtain. A second noteworthy feature of this scheme is its capacity to accommodate the scholastic theory of *suppositio*, with its concept of suppositional descent from 'All S is P' to 'This S is P' – for us, 'The

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1 The listing for the 'universal predicate' serves to determine the domain over which the subscripting variables are to range.

2 The analysis of a statement of the form '$[p_i] = [q_i]$' could be carried through in a manner parallel to Frege's analysis of 'the morning star = the evening star', and such an identity-statement would be classed as necessary if construed with reference to the extension (Fregean *Bedeutung*) of its terms, and as contingent if construed with reference to their intension (Fregean *Sinn*).

i-th S is P' – and from 'This S is P' to 'Some S is P'. As one commentator has remarked, this relationship among these three propositions escapes the approach customary in modern symbolic logic, because if 'All S is P' is rendered as '$(x)\,(Sx \supset Px)$' and 'Some S is P' as '$(\exists x)\,(Sx \,\&\, Px,)$' then these, unlike their medieval counterparts, will 'differ not just in quantification but also in internal structure'.[3] On the other hand, just such a parallelism is inherent in our present treatment.

A comparable, purely subscript-based, treatment of relations is also possible. Now instead of the linear list for a predicate we have a rectangular tabulation:

$$(r_i)\text{---}\colon\ (r_{i1})\text{---},\ (r_{i2})\text{---},\ldots,\ (r_{ij})\text{---},\ldots \quad i = 1, 2, \ldots,$$

where (r_{ij}) is the label for the j-th individual that stands in the relation R to $[r_i] = [r_{i0}]$. Concretely, if there are just four individuals a, b, c, d, and the relation R is such that a, b, and c each bears it to both the other two and to these only, and d bears it to nothing, then we have the (square) tabulation:[4]

$(r_{10})\ a\colon\ (r_{11})\,b,\ (r_{12})\,c,\ (r_{13})\,c,$
$(r_{20})\ b\colon\ (r_{21})\,a,\ (r_{22})\,c,\ (r_{23})\,c,$
$(r_{30})\ c\colon\ (r_{31})\,a,\ (r_{32})\,b,\ (r_{33})\,b,$
$(r_{40})\ c\colon\ (r_{41})\,a,\ (r_{42})\,b,\ (r_{43})\,b.$

Now the statement that R is anti-reflexive, '$\sim(\exists x)\,Rxx$', becomes:

$$\sim(\exists j)\,(i \neq 0 \,\&\, [r_{j0}] = [r_{ji}]).$$

The statement that R is symmetric, '$(\forall x)\,(\forall y)\,(Rxy \rightarrow Ryx)$', becomes:

$$(\forall j)\,[(\exists m)\,([r_{im}] = [r_{j0}]) \rightarrow (\exists n)\,([r_{jn}] = [r_{i0}])].$$

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[3] G.B. Matthews, 'Ockham's Supposition Theory and Modern Logic', *The Philosophical Review* **73** (1964) 91–99. On the concept of supposition and its treatment by the subscripting technique see E.A. Moody, *Truth and Consequence in Medieval Logic* (Amsterdam, 1953), pp. 35–36 and 51–52, and R.G. Turnbull, 'Ockham's Nominalistic Logic', *The New Scholasticism* **36** (1962) 313–329 (especially pp. 320–323).

[4] It is an essential feature of the present example that the array is a square one. We apply to relations the same redundancy-introducing 'stretching' device described above for predicates.

The statement '*Rba*' becomes '$(\exists j)\,([r_{20}]=[r_{1j}])$'. And in general all relational statements can be expressed with only our two postulated items of supra-propositional logical machinery: quantification over subscripts and individual-identity.

It is clear that an approach of this kind should excite much sympathy from anyone committed to a nominalistic or an extensionalist point of view.[5] However, the machinery that has been introduced is, *qua* machinery, strictly neutral as regards such philosophical commitments. For it would be possible to introduce into the picture non-existent individuals (non-designating singular terms) to serve as placeholders in labeled positions. With this – in principle perfectly feasible – step[6] (upon which we shall not elaborate here) one reintroduces all the complexities and perplexities which the extensionalists and nominalists seek to avoid.[7]

4. MANY-SORTED QUANTIFICATION

The root idea of the concept of many-sorted quantification is brought clearly to view by beginning with the problem of symbolizing such a statement as: 'Every soldier has a serial number'. This is usually treated in some such way as,

(1) $\qquad (\forall x)\,(Sx \supset (\exists y)\,Syx).$

where:

$Sx = x$ is a soldier
$Sxy = x$ is the serial number of y.

But this mode of treatment presupposes a rather oddly assorted universe of discourse bringing together such diverse bedfellows as men and numbers. A commonly employed alternative is to resort to *different styles of variables*, say

x, y, z, for men
k, m, n, for numbers

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[5] A nominalist would (though an extensionalist need not) restrict labeling index-sets to sets that are finite (or perhaps denumerable?).

[6] See T. Hailperin, and H. Leblanc, 'Nondesignating Singular Terms', *The Philosophical Review* **68** (1959) 239–243.

[7] See Chapter VII above. The substance of this section was presented in the author's article, 'Predicate Logic Without Predicates', *Logique et Analyse* **7** (1964) 101–103.

and then to reformulate (1) as,

(2) $(\forall x)(Sx \supset (\exists n)\, Snx)$

letting the differences in the appropriate sorts domain at issue be indicated by the different styles of variables. On the approach of *many sorted quantification*, however, only one single domain-neutral style of variable is employed and the domain-reference function is transferred to the quantifier by which the variable is bound. Thus we will introduce two different sorts of quantifiers:

$\forall_1, \exists_1$ for men
$\forall_2, \exists_2$ for numbers.

And now we reformulate (1) as:

(3) $(\forall_1 x)\,(Sx \supset (\exists_2 y)\, Syx)$.

The sharp difference among the items at issue are thus brought out in a clear and simple way.

On such an approach then, we do not base a system of quantificational logic on a single *universal* quantifier, $\forall$, with '$(\forall x)\ \phi x$' to be interpreted as 'for all values of the variable x ranging over the (all inclusive) domain of discourse D, it is the case that ϕ holds'. Rather, we have a series of *general* quantifiers, $\forall_i$ with $i = 1, 2, \ldots$, all subject to the convention that '$(\forall_i x)\ \phi x$' is to be interpreted as 'For all values of the variable x ranging over the (restricted) domain D_i, it is the case that ϕ holds'. The existential, or rather *special* quantifiers of this restricted sort would be introduced in the usual way, taking the general quantifier as primitive and introducing the special, by the definition:

$(\exists_i x)\ \phi x$ for '$\sim (\forall_i x) \sim \phi x$'.

In establishing in a formal way the well-defined character of many-sorted quantification systems, the present discussion shows that the status of the familiar one-sorted quantifier systems having a single universal quantifier is in no way sacrosanct. The customary *single mode of universal* quantification is here replaced by a *variety of modes of general quantification.*

This establishes a situation with which an analysis of existence with reference to quantification must come to terms. In particular, it shows clearly that 'existence' as relativized to general quantification is *not* a

monolithic and univocal concept. Again, the intimate relationship between a type of *variable* notation and a corresponding *domain of values*, which obtains in the usual one-sorted quantification theory – and also in Hao Wang's formulation of many-sorted logic – *is not* a necessary one, because, as the present treatment shows, variables can be restricted to playing simply the role of a notational cross-reference device which leaves the domain reference function to be absorbed by quantifiers. Thus, support should not be drawn from modern formal logic for the thesis that *existence*, even when regarded solely with reference to quantification, is a univocal concept.

W. V. O. Quine's dictum that 'To be is to be the value of a variable' is seen to appear in a thoroughly problematic light when the domain-reference function is shifted from the variables to the quantifiers, in the presence of alternative modes of existential quantification. It is as a matter of technical convenience that single-quantifier systems – with a monolithic mode of 'existence' – are so generally used, and not in response to any absolute requirements necessarily imposed by the inherent nature of the conceptual objects of logic.[8]

5. PLURALITY-QUANTIFICATION

We shall now consider the idea of *plurality-quantification*, introducing the special quantifier M, with '$(Mx)\ \phi x$' to be construed as: 'For *most* individuals x (in the non-empty domain of quantification D) it is the case that ϕx'. This is to be taken to say that the set of individuals (in D) for which ϕ obtains has a greater cardinality than the set for which this is false. (It is clear that the applicability of the notion is not restricted to finite universes.) This explanation fixes the semantical theory of this mode of quantification.

Some of the logical principles governing this quantifier are:

(1) $(\forall x)\, Px \supset (Mx)\, Px$

(2) $(Mx)\, Px \supset (\exists x)\, Px$

(3) $[(Mx)\, Px \,\&\, (Mx)\, Qx] \supset (\exists x)\, (Px \,\&\, Qx)$

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[8] This section draws substantially upon the author's paper, 'Many-sorted Quantification', *Proceedings of the 12th International Congress of Philosophy* (Venice, 1958); vol. 4, *Logic, Theory of Knowledge, Philosophy of Science, Philosophy of Language* (Firenze, 1960) 447–453.

(4) $(Mx)\,Px \supset \sim (Mx) \sim Px$

(5) $[(Mx)\,(Px \supset Qx) \,\&\, (\forall x)\,Px] \supset (Mx)\,Qx$

(6) $[(\forall x)\,(Px \supset Qx) \,\&\, (Mx)\,Px] \supset (Mx)\,Qx$.

It is clear that the converse of (4) does *not* obtain. For consider the case of a finite domain D with an even number of members, exactly half of which have P.

Notice also that all of these theses, as well as virtually all the ensuing remarks about the M-quantifiers are unaffected if '$(Mx)\,Px$' is construed (for finite universes) as 'For over 80% [instead of 50%] of all the individuals of D, we have it that P obtains'.

One interesting feature of the M-quantifier is that it is not self-commutative. Unlike the analogous case with universal or existential quantification, we do not have:

$$(Mx)\,(My)\,Rxy = (My)\,(Mx)\,Rxy.$$

This is easily shown by considering a universe of discourse of three members a, b, c, with relation R taken such that a bears R to nothing, b only to a and b, and c only to a and b.

It is readily seen that there is no way of providing a general definition of the M-quantifier in terms of $\forall$ and $\exists$.

It must be noted that '$(Mx)\,(Sx \supset Px)$' does not represent 'Most S is P' and indeed would not do so even if '$\supset$' were replaced by an implication-relation stronger than material implication. It is readily shown that there is no general way of providing a rendition of 'Most S is P' with the usual resources of quantificational logic, not even when these are supplemented by our plurality-quantifier M (or any other type of quantification, for that matter). Nevertheless the logic of the proportions of the forms 'most S is P' and 'Most S is not P' is an extremely simple matter. Indeed the syllogistic theory of such proportions can be treated by a validity-testing procedure which is but a minor elaboration upon the Venn-diagram technique.[9]

Consider the arguments:

All S's are P's	Most things are S's	Most Q's are S's
All parts of S's are parts of P's	Most things are P's	Most Q's are P's
	Some S's are P's	Some S's are P's

[9] See Chapter VII above.

It is often said that traditional logic is 'inadequate' because it cannot accommodate patently valid arguments like (1), and must therefore be supplemented by the resources of quantificational logic. But this holds equally true of modern quantificational logic itself, which cannot accommodate (2) until supplemented by something like our plurality-quantification. But even such expanded machinery cannot accommodate (3). To be sure, once the quantificational system is elaborated in some way to the point where *arithmetic* is possible, so that we can *count* the extension of a property P and compare the results of such countings, then this impotence is overcome. But this step requires us to move beyond the ill-defined borders of the province of logic, entering into that of mathematics proper.[10]

6. A QUANTIFICATIONAL TREATMENT OF MODALITY

The present section examines a hitherto little-explored approach to the logic of modality. This approach is based upon the idea of *possible though non-existent* objects. While some philosophers view this idea with distaste (in recent times most notably W. V. O. Quine), it has played a prominent role in the thought of others (the Stoics, Leibniz, Brentano and his school). In any case, this conception provides the basis for the ensuing considerations.

Our proposed quantificational construction of modality has the advantage – from the standpoint of the formal logician, at any rate – of reducing a (for him) relatively 'strange' discipline, viz. modal logic, to a relatively familiar one, viz. the logic of quantification. All of the formal machinery developed in the context of the latter branch of logic – proofs of completeness, consistency, etc. – can be brought to bear upon the former.

The root idea of our proposed construction of modality lies in con-

[10] The substance of this section was first presented in the author's note on 'Plurality-Quantification and Quasi-Categorical Propositions', *The Journal of Symbolic Logic* **27** (1962) 373–374. For an interesting indication of some advanced theorems that can be established with regard to plurality-quantifiers, and which put the logical theory into sharp contrast with that based upon the standard quantifiers $\forall$ and $\exists$ see David Kaplan, 'Rescher's Plurality-Quantification', *The Journal of Symbolic Logic* **31** (1966) 153–154; and *idem*, 'Generalized Plurality Quantification', *ibid.*, 154–155.

sidering the domain A of *actually existing objects* as subset of the domain P of *possible objects*. For the sake of convenience and exactness let us introduce some notational machinery. Let '$(\forall a)(\ldots a\ldots)$' and '$(\exists a)(\ldots a\ldots)$' represent universal and existential quantification over the (usual) domain A of actuals, and let '$(Aa)(\ldots a\ldots)$' and '$(Ea)(\ldots a\ldots)$' represent universal and existential quantification over the wider domain P of possibles.[11] (We shall return below to a further consideration of the idea of a 'domain of possibles'.)

Given this machinery, we can now present compactly the basic idea, the 'guiding intuition' for the proposed quantificational construction of modal propositions involving quantifiers. We propose to construe the statement that it is necessary that everything ϕ's as amounting to the thesis that every possible-object ϕ's; and analogously we shall construe the statement that it is possible that something ϕ's as amounting to the thesis that some possible-object ϕ's. We thus propose that

$$\Box(\forall x)\,\phi x$$

be regarded as amounting to

$$(Ax)\,\phi x$$

and that

$$\Diamond(\exists x)\,\phi x$$

amounts to: $(Ex)\,\phi x$.

It is at once clear that the usual duality relationships are preserved by this interpretation; so that, for example '$\sim\Box(x)\,\phi x$' is equivalent with '$\Diamond(Ex)\sim\phi x$'.

Among the formal requirements that a logical theory of modality-with-quantification is usually held as having to satisfy are the following:

(T1) $\vdash(\forall x)\Box\phi x\leftrightarrow\Box(\forall x)\,\phi x$

(T2) $\vdash(\exists x)\Box\phi x\rightarrow\Box(\exists x)\,\phi x$

(R1) $\dashv\Box(\exists x)\,\phi x\rightarrow(\exists x)\Box\phi x$

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[11] So that we of course have the bridging-rules

$$(Aa)(\ldots a\ldots)\rightarrow(\forall a)(\ldots a\ldots)$$
$$(Ea)(\ldots a\ldots)\rightarrow(\exists a)(\ldots a\ldots).$$

The resort to several styles of quantification can of course be circumvented by adopting only one (the widest) style of quantifier, and defining the others by means of it, imposing a suitable restrictive condition.

(T3) $\vdash (\forall x) \Diamond \phi x \leftrightarrow \Diamond (\forall x) \phi x$
(T4) $\vdash \Diamond (x) \phi x \rightarrow (\forall x) \Diamond \phi x$
(R2) $\dashv (\forall x) \Diamond \phi x \rightarrow \Diamond (\forall x) \phi x$.

Here '$\rightarrow$' represents entailment (or strict implication), '$\leftrightarrow$' represents mutual entailment (or strict equivalence), '$\vdash$' signalizes a thesis as acceptable (asserted), and '$\dashv$' signalizes a thesis as unacceptable (rejected).

If an adequate treatment of modality is to be given purely in terms of quantification, it must provide a quantificational construction of the modalities in such a way that these six requirements are met. It must thus provide a purely quantificational version of the following eight expressions:

(1) $\Box (\forall x) \phi x$
(2) $(\forall x) \Box \phi x$
(3) $(\exists x) \Box \phi x$
(4) $\Box (\exists x) \phi x$
(5) $\Diamond (\forall x) \phi x$
(6) $(\forall x) \Diamond \phi x$
(7) $\Diamond (\exists x) \phi x$
(8) $(\exists x) \Diamond \phi x$.

And furthermore, the purely quantificational rendition of these statement-forms must be such that the aforementioned requirements are satisfied, so that (1)$\leftrightarrow$(2), (3)$\rightarrow$(4) but not conversely, (5)$\rightarrow$(6) but not conversely, and (7)$\leftrightarrow$(8).

Returning now to the basic idea outlined above, we see at once that all of these requirements are satisfied by the following purely quantificational construction of quantified modal statements:

Modal Statement	Quantificational Construction
(1) $\Box (\forall x) \phi x$	$(\mathrm{A}x) \phi x$
(2) $(\forall x) \Box \phi x$	$(\mathrm{A}x) \phi x$
(3) $(\exists x) \Box \phi x$	$(\mathrm{A}x) \phi x$
(4) $\Box (\exists x) \phi x$	$(\forall x) \phi x$
(5) $\Diamond (\forall x) \phi x$	$(\exists x) \phi x$
(6) $(\forall x) \Diamond \phi x$	$(\mathrm{E}x) \phi x$
(7) $\Diamond (\exists x) \phi x$	$(\mathrm{E}x) \phi x$
(8) $(\exists x) \Diamond \phi x$	$(\mathrm{E}x) \phi x$

The reader can readily satisfy himself that each of the six requirements (T1)–(T3) and (R1)–(R3) is at once met by this purely quantificational construction of quantified modal statements.

But although this initial proposal is adequate to all requirements laid down so far, it has shortcoming that there are certain plausible *additional* requirements that if fails to satisfy. For it countenances two theses – viz. that (3) is no weaker than (1) = (2), and that (6) is no stronger than (7) = (8) – which, it could reasonably be argued, are unacceptable and must be rejected. It would thus be most plausible to add to the previous requirements two others:

(R3) $\dashv(\exists x)\Diamond\phi x \rightarrow (\forall x)\Diamond\phi x$

(R4) $\dashv(\exists x)\Box\phi x \rightarrow (\forall x)\Box\phi x$.

And if these two further requirements are added, our purely quantificational structure of modal statements at once collapses, because it fails to satisfy R3 and R4.

In the effort to extend our fundamentally quantificational construction of modality to a fully adequate theory capable of avoiding R3 and R4, let us take the somewhat bold – and yet to be justified – step of splitting the domain P of possible objects into two subdomains P_1 and P_2, the latter, P_2, including those elements of P which one is willing to regard as only *remotely* possible; the former, P_1, including the rest, i.e., those elements of P which one is willing to regard as *proximately* possible. We now obtain three groups of quantifiers:

(i) $(\exists a)$ and $(\forall a)$ over the domain of actual existents A
(ii) $(E_1 a)$ and $(A_1 a)$ over the domain $A \cup P_1$
(iii) $(E_2 a)$ and $(A_2 a)$ over the all-inclusive domain $A \cup P_1 \cup P_2$.

Here, then, two extraordinary modes of quantification are introduced, E_2 and A_2 corresponding to the old E and A; and E_1 and A_1 representing a new, intermediate mode of quantification, ranging not over *all* possible objects, but only over the *proximately* or *plausibly* possible objects.

The idea of *possible* as opposed to *actual* objects is anathema to some philosophers. And no doubt the idea of several types of possible objects, some more possible than others, would be viewed by them as weird nonsense. Let me try to motivate this idea.

It seems, first of all, that the idea of possible but not actual objects can most simply be introduced in terms of the factor of time. Thus in (i)–(iii) above we could let A be the set of *all presently actual* (i.e., currently existing) objects, P_1 the set of *all proximately actual* objects (i.e., objects in existence within some specified time τ of the present), and P_2 be the set of *all actual* objects (i.e., objects in existence at some time or other). This temporal approach has the merit of articulating the idea of *possible objects* without overstepping the bounds of that which is actual (at some time or other).

A second, perhaps no less plausible, way of differentiating between the proximately and the remotely possible is in terms of the frequently-discussed distinction between physical and logical possibility. Thus we could here regard as proximately possible an object whose description is compatible with the laws of nature (as golden goose eggs, for example, are not), and as remotely possible an object whose description is compatible with the laws of logic (as round squares, for example are not, but golden goose eggs are).

In either of these ways, and no doubt in others as well, sense can be made not only of the distinction between the actual and the possible, but even of that between the remotely and the proximately possible.

On the basis of this idea of a two-layer view of possible objects, we now re-apply the basic idea presented above as follows: We propose that $\Box(\forall x)\phi x$ be regarded as amounting to $(A_2x)\ \phi x$ and that $\Diamond(\exists x)\ \phi x$ be regarded as amounting to $(E_2x)\ \phi x$. Once again, then, we construe the statement that *necessarily* everything ϕ's as equivalent to the thesis that *every* possible-object ϕ's; and the statement that *possibly* something ϕ's as equivalent to the thesis that *some* possible-object ϕ's.

But we are now in a position to extend the purely quantificational construction of modal propositions in a more adequate way, as follows:

Modal Statement	Quantificational Construction
(1) $\Box(\forall x)\ \phi x$	$(A_2x)\ \phi x$
(2) $(\forall x)\Box\ \phi x$	$(A_2x)\ \phi x$
(3) $(\exists x)\Box\ \phi x$	$(A_1x)\ \phi x$
(4) $\Box(\exists x)\ \phi x$	$(\forall x)\ \phi x$
(5) $\Diamond(\forall x)\ \phi x$	$(\exists x)\ \phi x$

(6) $(\forall x) \Diamond \phi x$ $\quad$ $(E_1 x)\, \phi x$
(7) $\Diamond (\exists x)\, \phi x$ $\quad$ $(E_2 x)\, \phi x$
(8) $(\exists x) \Diamond \phi x$ $\quad$ $(E_2 x)\, \phi x$.

It is at once apparent that this mode of construction of quantified modal propositions satisfies not only the requirements T1–T4 and R1–R2 but also the added conditions R3–R4.[12]

It is not difficult to give a rationale for the just-indicated construction-scheme. Lines (1)–(2) and lines (7)–(8) are supported immediately by the considerations outlined above. Assuming for the moment that lines (4) and (5) have already been justified, lines (3) and (6) are readily supported. For (3) must be weaker than (1)–(2) but yet stronger than (4); and similarly (6) must be stronger than (5) but yet weaker than (7)–(8). It thus remains to consider the justification of lines (4) and (5). In this regard, it should be noted that given a justification of (5), (4) at once becomes justified in terms of (5) by negation-duality.

Let us finally consider the justification of line (5), i.e., the treatment of '$\Diamond (\forall x)\, \phi x$' as equivalent with '$(\exists x)\, \phi x$'. Assume first that '$(\exists x)\, \phi x$'. Then some things actually have ϕ. But then there is a 'possible world' – viz. that in which these things are the only things – in which all things have ϕ, so that: $\Diamond (\forall x)\, \phi x$. Conversely assume that: $\Diamond (\forall x)\, \phi x$. Now there is surely a sense of possibility (though perhaps a more than minimally strong one – as will be discussed below) such that we can only maintain that it is *possible* (in this sense) that *all* things have the property ϕ only when there is at least *one* thing that *actually* has ϕ.

The modal logic underlying this quantificational approach must, however, be very weak. Specifically, it cannot embody the (otherwise intrin-

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[12] Note that on the proposed construction iterated modalities are automatically defined for four cases, with the following results:

$\Box \Box (\exists x)\, \phi x \rightarrow \Box (\forall x)\, \phi x$
$\Diamond \Box (\exists x)\, \phi x \rightarrow \Diamond (\forall x)\, \phi x \leftrightarrow (\exists x)\, \phi x$
$\Box \Diamond (\forall x)\, \phi x \leftrightarrow \Box (\exists x)\, \phi x \leftrightarrow (\forall x)\, \phi x$
$\Diamond \Diamond (\forall x)\, \phi x \leftrightarrow \Diamond (\exists x)\, \phi x$

In view of the first equivalence it seems plausible that we should not want:

$\Box \Box P \leftrightarrow \Box P$

It is thus clear that the modal system at issue is to be weaker than Lewis' systems S4 or S5.

sically plausible) rules:

(R1) If $\vdash T$ (i.e., if T is a theorem), then $\vdash \Box T$.

(R2) If $\vdash T \supset S$, then $\vdash \Box T \supset \Box S$.

For it is a theorem of quantificational logic that:

$$\vdash (\exists x)\,[(\exists x)\,\phi x \supset \phi x].$$

But then by (R1),

$$\vdash \Box\,(\exists x)\,[(\exists x)\,\phi x \supset \phi x],$$

whence, by the postulated equivalence of '$\Box\,(\exists x)\,\phi x$' and '$(\forall x)\,\phi x$', we would have

$$\vdash (\forall x)\,[(\exists x)\,\phi x \supset \phi x].$$

But then we have it by quantificational logic that

$$\vdash (\exists x)\,\phi x \supset (\forall x)\,\phi x,$$

which is a consequence to be characterized in no milder terms than catastrophic. Similar unaccepted consequences can be shown to result from (R2). These rules must thus be ruled out for the modal logic at issue, which will therefore have to be a rather weak system (i.e., weaker than von Wright's system M).[13]

7. THE BARCAN INFERENCE

Consider an intensional modality such as $J(x)$ for 'John wishes that he owned x'. Notice that if we have a one-layer domain of actually existing objects (only), we cannot make the inference from '$J(a)$' to '$(\exists x)\,J(x)$', since John might, of course, wish for something that does not actually exist. Generally speaking, in the case of any modalized context about an 'apparent' individual in a modally qualified context '$\triangle(\ldots x\ldots)$' the question arises of whether we are entitled to draw the inference from this to '$(\exists x)\,\triangle(\ldots x\ldots)$'. Specifically, let us consider the following sort of inference by 'existential generalization'

$$\frac{\triangle \phi a}{(\exists x)\,\triangle \phi x}$$

[13] This section draws substantially upon the author's paper 'A Quantificational Treatment of Modality', *Logique et Analyse* 7 (1964) 34–42. I want to thank James Garson for helpful discussion of issues raised in the final paragraph of this section.

or rather, let us consider the somewhat stronger inference (its premiss being weaker):

$$\frac{\triangle (\exists x)\, \phi x}{(\exists x) \triangle \phi x}.$$

This may be called *the Barcan inference*, since it is analogous to the implication thesis

$$\Diamond [(\exists x)\, \phi x] \rightarrow (\exists x)\, [\Diamond \phi x]$$

which has become known as 'the Barcan formula' after Professor Ruth Barcan Marcus who first discussed it in a paper of 1948.

It is readily seen that this inference will fail with respect to necessity ($\Box$) on the quantificational constructions of modality presented in the previous section – as indeed it must in any adequate theory of modality (see (R1) on p. 173). But this (surely questionable) principle of inference will obtain with respect to possibility ($\Diamond$). This, however, is happily not an inevitable feature of all quantificational constructions of modality, as we shall see when discussing the temporal construction of modality on pp. 215–218 below.

8. TEMPORAL QUANTIFICATION AND MODALITY

We shall now return to the temporal construction of 'possible objects' suggested above. Let us introduce temporalized quantifiers over individuals, $(E_t a)$ and $(A_t a)$, such that:

'$(A_t x)\, \phi x$' means 'All x's existing at time t ϕ'
'$(E_t x)\, \phi x$' means 'Some x existing at time t ϕ'.

We shall also need (a) the temporal 'constant' n for 'now', (b) the constant τ delimiting 'proximate' time, and (c) (At) and (Et) as achronological quantifiers over times. We now have that:

'$(\forall x)\, \phi x$'	becomes	'$(Ax)\, \phi x$'
'$(\exists x)\, \phi x$'	becomes	'$(Ex)\, \phi x$'
'$(A_1 x)\, \phi x$'	becomes	'$(At)\, [(/n - t/ \leqslant \tau) \rightarrow (A_t x)\, \phi x]$'
'$(E_1 x)\, \phi x$'	becomes	'$(Et)\, [(/n - t/ \leqslant \tau)\, \&\, (E_t x)\, \phi x]$'
'$(A_2 x)\, \phi x$'	becomes	'$(At)\, (A_t x)\, \phi x$'
'$(E_2 x)\, \phi x$'	becomes	'$(Et)\, (E_t x)\, \phi x$'.

In the light of these constructions, we can re-examine our previous tabulation (pp. 176–177) to throw further light on the temporal construction of modality. Line (7) of that tabulation, for example, now tells us that only what has been actual for some x at some time or other is possible for some now-existing x. And similarly, line (5) has it that only what has been actual for some x at some time or other is possible for all now-existing x's. And similarly, line (5) has it that only that is *possible* for *all* now-existing x's which is *actual* for *some* now-existing x.

Just as the sense of 'possibility' at issue is a strong one (because '$\Diamond(\forall x)\ \phi x$' is true only when '$(\exists x)\ \phi x$' is), so the corresponding sense of necessity is also strong. For in view of line (4) we have $\Box(\exists x)\ \phi x \leftrightarrow (\forall x)\ \phi x$ (as a consequence it is, moreover, true that '$(\exists x)\ \phi x \rightarrow \Diamond(\forall x)\ \phi x$'). Our sense of 'necessity' is therefore such that it can be *necessary* that *some* individual has a property only if *all* individuals do *actually* have this property. *There are thus no 'differentially necessary properties'*, i.e., properties necessarily belonging to one individual but not to another. On the particular concept of necessity at issue, '$\Box(\exists x)\ \phi x \,\&\, \Box(\exists x) \sim \phi x$' is self-inconsistent. So long as even a single individual fails to have a given property, it is not necessary – in the strong sense of 'necessity' here at issue – that there be any individual that has this property. Thus on the approach at issue here, necessity is a mode of universality (viz., actuality in *all* instances), and possibility is a mode of particularity (viz., actuality in *some* instances). That this type of 'necessity' is only one among many alternatives that warrant serious considerations must of course be granted.

In the case of *abstract* entities, of numbers, for example, there are of course differentially necessary properties, so that the concept of necessity here at issue does not apply. (It is a patently necessary property of 5 – but not of 4 – to be a prime number.) And even in a *heterogeneous* class of concrete particulars this can be the case: it is a necessary proper property of Socrates (*qua* man) to be human, but not canine; and a necessary property of Fido (*qua* dog) to be canine, but not human. *Only if we take as our framework of discussion a homogenous group of concrete particulars – e.g., men – will the thesis that there are no differentially necessary properties become an eminently plausible one.* For whatever could reasonably be regarded as a necessary property of the man Socrates – being rational, mortal, animal, or the like – is patently also to be regarded as a necessary property of the man Caesar.

The point is that within a homogeneous group of concrete particulars, individuals cannot (by us mortals, at any rate) be differentiated through *essential* characteristics (*pace* Leibniz with his concept of the 'complete individual notion' of concrete particulars – which, however, is known only to God). The 'strong' concepts of (particular) necessity as 'that which is always exemplified' (in the sense that we have '$\Box(\exists x)\ \phi x \rightarrow (\forall x)\ \phi x$') and of (universal) possibility as 'that which is sometimes exemplified' (in the sense that we have '$(\exists x)\ \phi x \rightarrow \Diamond (\forall x)\ \phi x$') will thus be applicable in any such homogeneous group of concrete particulars.

In sum, then, it seems that among the many senses of necessity/possibility there is one (particularly strong) sense which conforms wholly to the requirements of our present discussion. There is, indeed, some reason to think that it was a concept of necessity/possibility akin to this one that was in the mind of Aristotle when he launched the discipline of modal logic on its long and eventful history.[14] But that is quite another story, and a very long one at that.

* * *

We may conclude our discussion with the observation that there are a substantial number of significant modes of quantification – 'existential' and 'universal' – that have nothing to do with the *existence of individuals*, and whose actuality casts doubt on the plausibility of the thesis that *To be is to be the value of a variable.*

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[14] See Jaakko Hintikka, 'Necessity, Universality and Time in Aristotle', *Ajatus* **20** (1957) 65–90, for some illuminating remarks about the relations between universality and necessity and between particularity and possibility in Aristotle.

CHAPTER XI

PROBABILITY LOGIC

1. BASIC CONCEPTIONS OF A PROPOSITIONAL PROBABILITY LOGIC

The leading idea around which the present discussion revolves is a concept of the *probability (or likelihood) of a statement.* It is presupposed that a function Pr is given which assigns to each statement p at issue some real-number value, to be indicated as $\Pr(p)$, which lies in the interval between 0 and 1 (inclusive). The function Pr is intended to provide a measure of the 'probability' of statements in the usual sense of that term. Specifically, it is supposed that this numerical measure function comports itself in a normal, 'well-behaved' way in satisfying the usual rules of the theory of statement-probabilities,[1] with the particular requirement that it meets *inter alia* the various conditions to be stipulated below.[2]

In the development of propositional probability-logic, our starting-point is a system of propositional logic of the familiar (classical) sort, supplemented with the alethic modalities of necessity and possibility ($\Box$ and $\Diamond$). We presuppose that the proposition variables at issue range over a domain of discourse D of definite, concrete statements, and that this domain D is *closed* under the relevant statement-generating operations, so that:

(1) Whenever p is an element of D,
so is $\sim p$.

(2) Whenever p and q are elements of D,
so is $p \vee q$.

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[1] Thus, for example, logically equivalent statements must be accorded the same probability-values.

[2] Several methods for securing a measure of the probability of statements in this fashion have been discussed in the literature cited in the References given at the end of the chapter. Of these, the (semantically grounded) method of Rudolf Carnap's important treatise on the *Logical Foundations of Probability* is the best known. However, no particular, specific method for the assignment of statement-probabilities need be assumed for our present purposes.

(3) Whenever p is an element of D,
so is $\Box p$.

To move now in the direction of obtaining probability-values for propositions we introduce the idea of a probability measure on the set D. A real-valued function $\Pr(p)$ defined for every statement p belonging to the domain of discourse D is a *probability measure* on D if the following three conditions are satisfied:

(PR1) $\Pr(p) \geqslant 0$, for every p in D.
(PR2) Whenever p is necessary (i.e., when $\Box p$), then $\Pr(p) = 1$.
(PR3) Whenever p and q are incompatible (i.e., when $\Box(\sim p \vee \sim q)$, then $\Pr(p \vee q) = \Pr(p) + \Pr(q)$.

These three conditions are in fact axioms for the probability-function, and assure that Pr obeys all of the usual rules for calculus of probability.[3]

Whenever $\Pr(q) \neq 0$, we can introduce a measure $\Pr(p/q)$ of the conditional probability of p on q:

$$\text{(CP)} \qquad \Pr(p/q) = \frac{\Pr(p \,\&\, q)}{\Pr(q)}.$$

This probability measure for statements is identical with one concept of a measure of 'degree of confirmation' among statements.[4] Several methods exist for the actual computation of numerical values of such a measure for statements within languages of various (generally quite simple) types, and Carnap's *Logical Foundations for Probability* should be consulted for details and for references to the literature. However, no details regarding specific Pr measures will be required here. For our present purposes, the notion of a probability measure for statements will be used in only an abstract, general way, in order to simplify problems of explanation and exposition.

We may look upon the probability-value of a proposition as a second, somewhat nonstandard type of truth-value. From this standpoint, we can take the view that with any statement p there is associated a somewhat

[3] See P. R. Halmos, 'The Foundations of Probability', *American Mathematical Monthly* **51** (1944) 493–510.

[4] On this subject the reader is referred to R. Carnap's *Logical Foundations of Probability* (Chicago, 1950).

odd, two-place truth-value made up of an ordered pair of values, (V(p), Pr(p)), where V(p) is the ordinary truth-value of p (namely 0 for false and 1 for true), and where Pr(p) is the numerical probability of p in the sense described above. We take these two quantities to be *independent* of each other *except for* the following stipulation: If $\Pr(p)=1$, then $V(p)$ must also be 1, and if $\Pr(p)=0$, then $V(p)$ must also be 0. (We thus assume that we are dealing with a *finite* possibility-space, since only then does a probability of 1 assure truth, and a probability of 0 assure falsity.)

For compound statements, the V-value is *determined* in the usual truth-functional way based upon the familiar truth-tables. Further – since the probability values are postulated to be 'well behaved' in conforming to the usual principles of probability calculation – we are also in a position to *delimit* the (already given) Pr-value of a compound statement in terms of the Pr-values of its constituents. Several basic facts obtaining in virtue of the postulated characteristics of Pr are embodied in the following rules:

Statement of Form	V-value	Pr-value
(1) $\sim p$	$1 - V(p)$	$1 - \Pr(p)$
(2) $p \vee q$	$\min[1, V(p) + V(q)]$	$\Pr(p) + \Pr(q)$, if p and q are mutually exclusive; and otherwise a certain quantity Q such that $Q \leqslant \Pr(p) + \Pr(q)$, but $Q \geqslant \Pr(p)$ and $Q \geqslant \Pr(q)$.[5]

Since the remaining operations of propositional logic are definable in terms of the foregoing two, derivative rules for delimiting their V-values and Pr-values can be obtained. Among others, this yields the following rules:

[5] It must be remembered that we are not claiming to *characterize* Pr at this juncture, but simply to note some of its traits which derive from its previously fixed character as a *given* measure over the domain D that conforms to the usual rules of a probability-calculus for statements.

Statement of Form	V-value	Pr-value
(3) $p \& q$	$V(p) + V(q) - V(p \vee q)$	$Pr(p) + Pr(q) - Pr(p \vee q)$. [Note that this is a quantity Q such that $Q < Pr(p)$ and $Q < Pr(q)$.]
(4) $p \supset q$	$\min[1, 1 - V(p) + V(q)]$	As dictated by Rules 1 and 2 above for: $\sim p \vee q$.

It is thus clear that our assignment of two-place truth-values is *not* truth-functional in the familiar sense that the truth-value of a compound proposition is determined by that of its constituents. Thus if the two-place truth-value of p_1, is $(1, \frac{1}{2})$ and that of p_2 is $(0, \frac{1}{2})$, then the two-place truth-value of $p_1 \& p_2$ will in general be $(0, x)$ where $0 < x < \frac{1}{2}$, but it will certainly have to be $(0, 0)$ in the special case when p_2 is $\sim p_1$.

2. THE LIKELIHOOD MODALITY

The root idea of a *likelihood modality* is most simply stated in terms of absolute (i.e., unconditional) modalities. We introduce a new alethic modality, symbolized by '$\triangle$' where $\triangle p$ is intended to assert that p is probable (likely). This probability modality is related to those of necessity and of possibility by the relationships: (1) $\Box p \rightarrow \triangle p$ and (2) $\triangle p \rightarrow \Diamond p$. However, this (unconditional) probability modality is not definable in terms of the other modalities, nor are they definable in terms of it.[6]

In terms of a measure of statement probabilities, we can now interpret '$\triangle p$' as answering, in effect, to the condition that $Pr(p) > \frac{1}{2}$. Since $\Box p$ entails $Pr(p) = 1$ (though not necessarily conversely), we have $\Box p \rightarrow \triangle p$ and $\triangle p \rightarrow \Diamond p$, as desired. We further have $\triangle p \rightarrow \sim \triangle \sim p$ (though not necessarily conversely, since $\sim \triangle \sim p$ amounts to $Pr(p) \geq \frac{1}{2}$, with the weak inequality).

[6] This is readily shown by means of the probabilistic interpretation of modalities now to be discussed.

Consequences of particular interest arise when we move into the sphere of conditional modalities. Here we introduce the conditional likelihood modality $\triangle(p/q)$, to be read, and interpreted, as 'p is probable (likely), given q'. In terms of our conception of statement probabilities, this is to be taken to amount to the requirement that the conditional probability of p and q is greater than one-half, i.e., $\Pr(p/q) > \frac{1}{2}$. This conditional likelihood of modality is strictly analogous to the conditional modality of necessity, which we may introduce by the rule that $\Box(p/q)$ is to be taken to amount to the requirement that the conditional probability of p given q is one, i.e., $\Pr(p/q) = 1$.

As the characteristic principle of inference governing the modality of conditional probability we have the rule:

$$(\triangle 1) \qquad [\triangle(p/q) \,\&\, \Box q] \rightarrow \triangle p.$$

The rationalization of this rule in terms of the intended meaning of the concepts involved is as follows: If $\triangle(p/q)$, then $\Pr(p/q) > \frac{1}{2}$. Therefore $\Pr(p\,\&\,q) > \frac{1}{2}\Pr(q)$. But if q is necessary, then $\Pr(q) = 1$. Thus $\Pr(p) > \Pr(p\,\&\,q) > \frac{1}{2}$. Q.E.D.

The rule $(\triangle 1)$ is a strict analogue of corresponding the following rule of modal logic:

$$[\Diamond(q \rightarrow p) \,\&\, \Box q] \rightarrow \Diamond p.$$

Some further examples of rules for the (conditional) probability are as follows:

$$(\triangle 2) \qquad \sim \Diamond q \rightarrow \sim \triangle(q/p)$$
$$(\triangle 3) \qquad \triangle(p/q) \rightarrow \sim \triangle(\sim p/q)$$
$$(\triangle 4) \qquad [\triangle p \,\&\, \Box(q/p)] \rightarrow \triangle q$$
$$(\triangle 5) \qquad [\Diamond p \,\&\, \triangle(q/p)] \rightarrow \Diamond(p \,\&\, q)$$
$$(\triangle 6) \qquad [\triangle(p/q) \,\&\, \Box(r \rightarrow q)] \rightarrow \triangle(p/r).$$

We shall not attempt here to give a complete axiomatization of the rules for the probability-modality $\triangle$. Rather we shall content ourselves with setting forth a *criterion of acceptability* for such rules, akin to the truth-tables of propositional logic. Consider the following set of stipulations:

The indicated proposition is true (or not) according as	The corresponding condition stipulated below is (or is not) satisfied
1. $\Diamond p$	1. $\Pr(p) \neq 0$
2. $\Box p$	2. $\Pr(p) = 1$
3. $\triangle p$	3. $\Pr(p) > \frac{1}{2}$
4. $\Box(p/q)$	4. $\Pr(p/q) = 1$
5. $\triangle(p/q)$	5. $\Pr(p/q) > \frac{1}{2}$

A formula will be an acceptable thesis of the system of modal logic now under construction if it becomes *a truth of the Pr-calculus for any assignment of Pr-values to its propositional constituents.* Thus, for example, ($\triangle$4), on applying the above-indicated interpretation schema, comes to:

$$[\Pr(p) > \tfrac{1}{2} \,\&\, \Pr(q/p) = 1] \rightarrow \Pr(q) > \tfrac{1}{2}.$$

This is readily verified – via the definition of conditional probability – to be a general truth of the calculus of probabilities (and is *generically* true, regardless of the specific Pr-values that are at issue). This approval exemplifies the testing-procedure for the acceptability of theses in our system for the likelihood-modality.

3. A PROBABILISTIC APPROACH TO MODAL LOGIC

The aim of this section is to present an approach to modal propositional logic based on the concept of probability. It will be shown that C. I. Lewis' system of strict implication S5 can be regarded, from an appropriate point of view, as being a propositional probability logic.[7] Apart from whatever intrinsic interest such a discussion may have, it gains added significance from the fact that it provides a bridge linking the two disciplines of many-valued logics and modal logics.

[7] The present section draws upon ideas originally developed in the author's paper, 'A Probabilistic Approach to Modal Logic', *Acta Philosophica Fennica*, fasc. **16** (1963) 215–226.

Let us again consider the probabilistic versions of familiar alethic modalities. $\Box(p)$ for 'p is necessary' and $\Diamond(p)$ for 'p is possible', the former being taken as primitive, and the latter as defined in terms of it by the definition:

$$\Diamond p \text{ for } \sim \Box(\sim p).$$

Furthermore, an operation of strict implication will be introduced by the definition:

$$p \dashrightarrow q \text{ for } \Box(p \supset q).$$

The calculation of the two-place truth-values for statements involving the modal operators is to be achieved by the postulation of the following rules:

(i) That a statement of the type $\Box(p)$ is to be true (i.e., have V-value 1) if and only if $\Pr(p) = 1$.

(ii) That a statement of the type $\Box(p)$ is to be capable of assuming only the Pr-values 0 and 1.

As a consequence, the V-values and Pr-values of statements involving modal operators are governed by the following rules:

Statement of Form	V-value	Pr-value
(5) $\Box(p)$	Same as the Pr-value	$\begin{cases} 1 \text{ if } \Pr(p) = 1 \\ 0 \text{ otherwise} \end{cases}$
(6) $\Diamond(p)$	Same as the Pr-value	$\begin{cases} 0 \text{ if } \Pr(p) = 0 \\ 1 \text{ otherwise} \end{cases}$
(7) $p \dashrightarrow q$	(As determined for $\Box(p \supset q)$.)	(As determined for $\Box(p \supset q)$.)

It should be noted that, in view of the relative independence of the V-values and Pr-values, the truth – i.e., having V-value of 1 – of $\Diamond p$ is perfectly compatible with that of $\Diamond(\sim p)$.

The remainder of this chapter will be concerned with the development of the formal theory of the logic of propositional modality based upon these conceptions.

* * *

We shall characterize a statement schema of our system of modal logic as an *M-tautology, if the two-place truth-value of this schema is (1,1) identically, for any assignment of two-place truth-values to its propositional components.* To indicate that the statement schema s is an M-tautology we shall write:

$$\vdash_M s .$$

Since the V-value of any statement having the Pr-value of 1 must *ipso facto* be 1, also, it suffices to establish a statement schema as an M-tautology to show that its Pr-value is identically 1. We thus have the fundamental rule:

Rule 0. $\vdash_M s$ if and only if $\mathrm{Pr}(s) \equiv 1$, for all Pr-value assignments to the variables of s.[8,9]

Since a statement of the type $\Box(p)$ is true – i.e., has V-value 1 – whenever $\mathrm{Pr}(p) = 1$, we have the general rule:

If $\vdash_M s$, then $\Box s'$, for any substitution-instance s' of s.

We next establish several further important rules that govern M-tautologies. The value of these rules is two-fold: they throw light on the

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[8] In this rule '$\equiv$' represents the familiar algebraic notation for an identity, rather than material equivalence.

[9] In view of Rule 0, it might be thought that – rather than introducing a two-place-truth value, and then defining M-tautology in its terms – the best procedure would be to define M-tautologousness directly by Rule 0. Although this tactic would indeed provide an adequate basis for the whole of the present discussion, I have chosen to adhere to two-place truth values for the following reasons. Firstly this machinery seems to me to have certain didactic advantages in keeping the notion of statement probabilities linked with the familiar resource of truth-values. Secondly it makes semantical considerations clearer and more explicit. Finally, it makes possible an extension of the considerations of the present discussion to such existential axioms as Lewis's B9, or

$$(\exists p)(p \,\&\, \sim \Box p),$$

which could not be dealt with without the two-place truth-values or some similar mechanism.

concept of M-tautology itself, and they furnish requisite lemmas for the ensuing discussion.

Rule 1. If $\mid M \vdash s_1 \supset s_2$, then $\Pr(s_1) \leqslant \Pr(s_2)$.
Proof: If $\mid M \vdash s_1 \supset s_2$, then $\Pr(\sim s_1 \vee s_2) = 1$.
Then $1 \leqslant \Pr(\sim s_1) + \Pr(s_2) = 1 - \Pr(s_1) + \Pr(s_2)$.
Consequently $\Pr(s_1) \leqslant \Pr(s_2)$. Q.E.D.

Rule 2. If $\mid M \vdash s_1 \supset s_2$, and $\mid M \vdash s_1$, then $\mid M \vdash s_2$.
Proof: This is an immediate consequence of Rule 1.

Rule 3. $\mid M \vdash s_1 \,\&\, s_2$ iff (if and only if) both $\mid M \vdash s_1$ and $\mid M \vdash s_2$.
Proof: (i) Assume $\mid M \vdash s_1 \,\&\, s_2$, i.e. $\Pr(s_1 \,\&\, s_2) = 1$.
But $\Pr(s_1 \,\&\, s_2)$ is $\leqslant \Pr(s_1)$ and also $\leqslant \Pr(s_2)$, so that both of these are 1.
(ii) Assume $\mid M \vdash s_1$ and $\mid M \vdash s_2$, so that $\Pr(s_1) = \Pr(s_2) = 1$.
But $\Pr(s_1 \,\&\, s_2) = \Pr(s_2) + \Pr(s_1) - \Pr(s_1 \vee s_2) = 2 - \Pr(s_1 \,\&\, s_2)$.
Hence $\Pr(s_1 \,\&\, s_2)$ must be 1, and so $\mid M \vdash s_1 \,\&\, s_2$. Thus Q.E.D.

Rule 4. If $\mid M \vdash s_1 \equiv s_2$, then $\Pr(s_1) = \Pr(s_2)$.
Proof: This rule is a consequence of Rules 1 and 3.

It follows by Rules 2 and 4 that the rules of inference of *modus ponens* and substitution of (demonstrable) equivalents are M-tautology preserving, i.e., they yield M-tautologies as conclusions when applied to M-tautologies as premisses. It follows further that whenever some set of axioms are M-tautologies all of the theorems derivable by means of these rules of inference must be M-tautologies as well.

We have introduced a species of strict implication into our modal logic by means of the familiar definition:

$$p \prec q \text{ for } \Box(p \supset q).$$

Let it be recalled that we have already claimed the result that, for any theorem t of (standard, two-valued) propositional logic, $\mid M \vdash t$. This assures the establishment of:

Rule 5. For any theorem of propositional logic of the form $s_1 \supset s_2$ $\mid M \vdash s_1 \prec s_2$.

The strict-implication relationship is also governed by the rule:

Rule 6. $\mid M \vdash s_1 \prec s_2$ iff $\Pr(\sim s_1 \vee s_2) \equiv 1$.
Proof: Obvious.

We shall make frequent use of these rules in establishing strict-implication statements as M-tautologies.

* * *

The next order of business is to exhibit some further concrete examples of statement schemata that represent M-tautologies:

In view of Rule 5 above, we immediately have:

(M1) $(p \,\&\, q) \dashv\prec (q \,\&\, p)$
(M2) $(p \,\&\, q) \dashv\prec p$
(M3) $(p \,\&\, q) \dashv\prec q$
(M4) $p \dashv\prec (p \vee q)$
(M5) $[p \,\&\, (q \,\&\, r)] \dashv\prec [q \,\&\, (p \,\&\, r)]$
(M6) $[(p \,\&\, q) \,\&\, r] \dashv\prec [p \,\&\, (q \,\&\, r)]$
(M7) $p \dashv\prec \sim\sim p.$

All seven of these schemata are shown to be M-tautologies by the fact that they fall directly into the range of application of Rule 5.

Somewhat greater interest perhaps attaches to the following M-tautologies:

(M8) $\Box p \dashv\prec p$

Proof: We must show that $Q = \Pr(\sim \Box p \vee p) = 1$.

But we know that $Q \geqslant \begin{cases} (1)\ \Pr(\sim \Box p) = 1 - \Pr(\Box p) \\ (2)\ \Pr(p) \end{cases}$.

We proceed to a proof by cases.

Case 1. $\Pr(p) = 1$, then Q.E.D. by (2).

Case 2. $\Pr(p) \neq 1$, then $\Pr(\Box p) = 0$, so that Q.E.D. by (1).

(M9) $[(p \dashv\prec q) \,\&\, (q \dashv\prec r)] \dashv\prec (p \dashv\prec r)$

Proof: We must show that $Q = \Pr[\sim \Box (p \supset q) \vee \sim \Box (q \supset r) \vee \Box (p \supset r)] = 1$.

Now $Q \geqslant \begin{cases} (1)\ \Pr[\sim \Box (p \supset q)] = 1 - \Pr[\Box (p \supset q)] \\ (2)\ \Pr[\sim \Box (q \supset r)] = 1 - \Pr[\Box (q \supset r)] \\ (3)\ \Pr[\Box (p \supset r)] \end{cases}$

But by (1) and (2), $Q = 1$ unless $\Pr[\Box (p \supset q)] = \Pr[\Box (q \supset r)] = 1$.

Suppose then that we have $\Pr(p \supset q) = \Pr(q \supset r) = 1$.

Then $\Pr[(p \supset q) \& (q \supset r)] = 1$.
Now by Rule 1, $\Pr[(p \supset q) \& (q \supset r)] \leqslant \Pr(p \supset r)$, so that $\Pr(p \supset r) = 1$.
But then $Q = 1$ by (3). So Q.E.D.

(M10) $[p \& (p \prec q)] \prec q$

Proof: We must show that:
$\Pr[\sim p \vee \sim \Box(p \supset q) \vee q] = 1$.
But the left-hand quantity equals $\Pr[\sim \Box(p \supset q) \vee (p \supset q)]$, and this is seen to be 1 by use of (M8).

(M11) $\sim \Box(\sim p) \prec \Box(\sim \Box(\sim p))$

Proof: We must show that $Q = \Pr[\Box(\sim p) \vee \Box(\sim \Box(\sim p))] = 1$.

Now $Q \geqslant$ (1) $\Pr[\Box(\sim p)]$
(2) $\Pr[\Box(\sim \Box(\sim p))]$

But by (1), $Q = 1$ unless $\Pr[\Box(\sim p)] = 0$. And then $\Pr[\sim(\Box(\sim p))] = 1$. And then $Q = 1$ by (2). So Q.E.D.

(M12) $(p \prec q) \prec (\sim q \prec \sim p)$

Proof: We must show that
$Q = \Pr[\sim \Box(p \supset q) \vee \Box(\sim q \supset \sim p)] = 1$.

Now $Q \geqslant$ (1) $\Pr[\sim \Box(p \supset q)] = 1 - \Pr[\Box(p \supset q)]$
(2) $\Pr[\Box(\sim q \supset \sim p)] = \Pr[\Box(q \vee \sim p)]$

But by (1), $Q = 1$ unless $\Pr(p \supset q) = \Pr(\sim p \vee q) = 1$.
And then Q is 1 by (2). So Q.E.D.

(M13) $[\Box p \& \Box q] \prec \Box(p \& q)$

Proof: We must show that
$Q = \Pr[\sim \Box p \vee \sim \Box q \vee \Box(p \& q)] = 1$.

Now $Q \geqslant$ (1) $\Pr[\sim \Box p] = 1 - \Pr[\Box p]$
(2) $\Pr[\sim \Box q] = 1 - \Pr[\Box q]$
(3) $\Pr[\Box(p \& q)]$

Case 1. $\Pr(\Box p) = 0$ or $\Pr(\Box q) = 0$. Then Q.E.D. by (1) or (2)
Case 2. $\Pr(\Box p) \neq 0$ and $\Pr(\Box q) \neq 0$.
Then $\Pr(p) = \Pr(q) = 1$.
Therefore $\Pr(p \& q) = 1 + 1 - \Pr(p \vee q) = 1$.
So Q.E.D. by (3).

(M14) $\Box p \prec \Box(\Box p)$

Proof: We must show that $Q = \Pr[\sim\Box p \vee \Box(\Box p)] = 1$.

Now $Q \geqslant$ (1) $\Pr[\sim\Box(p)] = 1 - \Pr(\Box p)$
(2) $\Pr[\Box(\Box p)]$

But by (1), Q is 1 unless $\Pr[\Box(p)] = 1$.
But then Q is 1 by (2). So Q.E.D.

(M15) $\Diamond(\Box p) \dashv\prec \Box p$
Proof: Analogous with that of (M14).

(M16) $\Diamond p \dashv\prec \Box(\Diamond p)$
Proof: Analogous with that of (M14).

(M17) $\Diamond(\Diamond p) \dashv\prec \Diamond p$
Proof: Analogous with that of (M14).

It may be remarked that in virtue of (M14)–(M17), we have:

(i) $XY...Z(\Box(p)) \rightarrowtail\!\!\dashv\prec \Box(p)$, where X, Y, ..., Z represent modalities of type $\Box$ or $\Diamond$.

And analogously,

(ii) $XY...Z(\Diamond(p)) \rightarrowtail\!\!\dashv\prec \Diamond(p)$.

The 'inmost' modality alone is the determining one.

* * *

We had already succeeded in showing that a system of modal logic (with $\dashv\prec$ as its sole mode of implication connective) based upon the acceptance of all *M*-tautologies embraces the whole of Lewis' modal system S5. For S5 is derived from the following eight theses as axioms: (M1), (M2), (M4), (M6), (M7), (M9), (M10), and (M11). And we have shown not only that all of these theses are *M*-tautologies, but also that the rules of inference are *M*-tautology preserving. Thus it is established that *all theorems of* S5 *are M-tautologies.*

This observation naturally raises the converse question, the question of completeness: *What axiom-system of modal propositional logic, if any, is (demonstrably) capable of yielding all M-tautologies as theorems*? Specifically, does S5 furnish an adequate axiomatic basis for the whole family of the *M*-tautologies with modal operators?

This question is to be answered affirmatively. It has been shown by Paul Henle[10] that S5 is mathematically equivalent to Boolean algebra, when (i) this algebra is interpreted for propositions, and (ii) the propositional operator $\Box$ is given a Boolean interpretation by the stipulation that $\Box(p)$ is to amount to $p = V$ (where 'V' represents the universal element of the Boolean algebra). The phrase 'mathematically equivalent' is here to be construed as asserting (*inter alia*) that:

> Every theorem of Boolean algebra leads to a theorem of S5 upon the indicated propositional reinterpretation.

Now suppose that a certain statement schema s of our modal logic is an M-tautology. Then s must necessarily be, in its Boolean reinterpretation, a theorem of Boolean algebra. This is so because, it is readily seen that, by the character of the rules of probability-calculation, *s can only take on the* Pr-*value* 1 *if (in Boolean language)* $s = V$.[11] Consequently, it thus follows almost immediately from the cited result of Henle's that our assumed M-tautology s must be a theorem of S5.

We have therefore established the completeness of S5 in the manner indicated in the heading of the present section, i.e., in the precise sense that S5 furnishes an axiomatic basis for exactly the family of M-tautologies.[12]

(As regards formulas in which modal operators do not occur, it is readily shown that all the theorems of classical propositional logic (PC)

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[10] See Appendix II of C.I. Lewis and C.H. Langford, *Symbolic Logic* (New York, 1932).

[11] At this point too it is important that the possibility-space for the Pr measure is finite, since otherwise the possession by s of Pr-measure of 1 would not guarantee the Boolean identity of s with the universal element V.

[12] Since the days of J. M. Keynes there has existed in the literature a concept of probability that is merely *qualitative* or comparative rather than full-bloodedly *quantitative*. In particular, see Leonard J. Savage, *The Foundations of Statistics* (John Wiley and Sons, New York and London 1954). This gives an axiomatic basis for a qualitative concept of probability. Although the present discussion has been based on a quantitative measure of probability, it has actually made use of only a few of the grosser characteristics of such a measure. Indeed, all of the results presented would still be forthcoming if this qualitative probability concept were taken for its basis, and the condition of 'having probability 1' were replaced by the condition 'having greatest possible comparative probability'. (I owe the substance of this footnote to Professor Patrick Suppes.)

are *M*-tautologies. For it is easily checked that some standard set of axioms for PC are *M*-tautologous, and we have already seen that the usual rules of inference – i.e., *modus ponens* and substitution – are such as to yield *M*-tautologous conclusions when applied to *M*-tautologies as premisses.)

* * *

In concluding, let me first make a brief retrospect. The leading idea of the present section has been a probabilistic construction of propositional modality, based on the idea that a necessary proposition may be construed as one whose probability-value is 1. On the basis of this approach, we have introduced the conception of an *M*-tautology, i.e. (essentially) a statement schema whose probability-value is *uniformly* or identically 1, for all assignments of probability-values to its constituents. We have been able to show that Lewis' modal system S5 provides a complete axiomatization for the 'tautologies' in this probabilistic sense.[13]

It would seem that some such probabilistic (and thus both infinite-valued and non-truth-functional) mode of 'many-valued logic' is as close as one can come to a fusion between strict implication and many-valued logic in view of Dugundji's theorem to the effect that there can be no finite characteristic matrix for any one of the systems S1–S5.

Our findings seem to afford yet another significant point of support for the claims of S5 to be regarded as a very 'natural' system of modal propositional logic. For viewed from an appropriate standpoint, the modal system S5 is seen to represent an axiomatic formalization of a propositional probability logic. By thus providing a construction of modality in probabilistic terms, our findings put modal propositional logic, at any rate in the particular guise of S5, into an interesting and illuminating perspective.

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[13] See James Dugundji's 'Note on a Property of Matrices for Lewis and Langford's Calculi of Propositions', *The Journal of Symbolic Logic* **5** (1940) 150–151.

CHAPTER XII

CHRONOLOGICAL LOGIC

1. BACKGROUND

The object of chronological logic – 'tense logic' or 'change logic' as it has also been called by various authors – is to systematize reasoning with propositions that have a temporalized copula. Such propositions do not involve the timeless 'is' (or 'are') of the mathematicians' '3 is a prime', but rather envisage an explicitly temporal condition: 'Bob *is* sitting', 'Robert *was* present', 'Mary *will have been* informed'. In this area, we have to do with statements involving 'time talk' in which some essential reference to the Before-After relationship or the Past-Present-Future relationship is at issue, and the ideas of succession change and constancy enter in. Chronological logic seeks to provide the linguistic and inferential apparatus for exact discourse and rigorous reasoning in this sphere.

The logical theory of such time-related propositions is of substantial interest because these explicitly temporal considerations arise in a wide variety of philosophically relevant contexts. Apart from their obvious significance for the analysis of tensed discourse, they are germane to various interests of philosophers of science – the structure of time, the analysis of temporal relations, (e.g. of temporal conjunction and contiguity, important for the analysis of principles of causal inquiry such as Mill's methods), and the characterization of natural processes, among others. Mediated through the linkage of temporally sequential processes, they come to have a bearing on the concept of *sets of instructions*, and so have a bearing upon the logical theory of commands, thus entering the purview of ethics through the command theory of moral imperatives. Moreover, they are of interest to the logician, both in their own right, and because of their involvement with the theory of modality, *via* the chronologized conception of modality along lines to be explained shortly.

A few words must be devoted to the historical background. Chronological logic originates with the Megarians who had a conception about modality according to which the ACTUAL is that which is realized *now*,

the POSSIBLE that which is realized *at some time or other*, and the NECESSARY that which is realized *at all times*. These Megarian ideas can be found also in Aristotle, together with another temporalized sense of necessity according to which certain possibilities are *possible* prior to the event, *actual* then, and *necessary* thereafter, so that their modal status is not omnitemporal (as on the Megarian concept), but changes over time.[1] Mention should also be made of the Diodorean concept of implication (named after the Stoic logician Diodorus Cronus) which (for example) has it that the conditional 'If the sun has risen, it is daytime' is to be given the temporal construction 'All times after the sun has risen are times when it is daytime'.[2] The Arabic logician Avicenna (d. 980) treated this chronological conception of implication in the framework of a general theory of categorical propositions of a temporalized type,[3] and also developed considerably the Megarian–Stoic theory of temporal modalities.[4]

The medieval Latin schoolmen also taught a temporalized theory of categorical propositions in terms of their doctrine of the 'ampliation' of terms, articulating such rules as the following:

> Every term having supposition, as subject, with respect to a verb of past time, is ampliated to stand for that which exists or for that which has existed.
>
> Some A was a $B = (\exists x)\,[\{R_n(Ax) \vee R_p(Ax)\} \,\&\, R_p(Bx)]$.
> Every A was a $B = (\forall x)\,[\{R_n(Ax) \vee R_p(Ax)\} \supset R_p(Bx)]$.
>
> Every term having supposition, as subject, with respect to a verb of future time, is ampliated to stand for that which exists or for that which will exist.
>
> Some A will be a $B = (\exists x)\,[\{R_n(Ax) \vee R_f(Ax)\} \,\&\, R_f(Bx)]$.
> Every A will be a $B = (\forall x)\,[\{R_n(Ax) \vee R_f(Ax)\} \supset R_f(Bx)]$.[5]

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1 The Stoic conception of temporal modality is yet another cognate, according to which the *possible* is that which is realized at *some* time in the present-or-future and the *necessary* that which is realized at *all* such times.

2 The Megarian and Stoic theories are discussed in RESCHER (1967a). (For references of this sort see the Bibliography at the end of this chapter.)

3 See RESCHER (1963) for details.

4 See RESCHER (1966b).

5 MOODY (1953), p. 56. I have altered Moody's notation.

Here we take '$R_n(q)$' to mean 'q is realized now', '$R_p(q)$' to mean 'q was realized in the past' and '$R_f(q)$' to mean 'q will be realized in the future'.

The stimulus for a revival of chronological logic in recent times can be credited to three sources: the study of the historical materials outlined in the preceding paragraph, the logical analysis of grammatical tenses by Hans Reichenbach[6], and (above all) the endeavor by the Polish logician Jerzy Łoś to devise a chronological logic for the study of problems in the philosophy of science, especially Mill's methods of inductive reasoning.[7] Łoś's ideas were considerably developed and extended by Arthur N. Prior,[8] and following Prior by Nicholas Rescher.[9] Independently of Prior (and apparently of Łoś as well) is the recent development by G. H. von Wright of a (substantially weaker) system of a chronological 'logic of change',[10] which has been extended in various directions by several logicians. A general picture of recent developments can be gleaned from the Bibliography presented at the conclusion of this chapter.

2. PRELIMINARY DISTINCTIONS

A. *The Temporal Equivocality of IS*

Logicians have frequently dwelt upon the equivocation of 'is' as between the '*is* of identity' on the one hand, and the '*is* of predication' on the other. The temporal equivocation of 'is' has, however, been little heeded. Yet it is quite clear that there are several very distinct possibilities:

(i) The 'atemporal *is*' that means 'is *timelessly*'. ('Three *is* a prime number'.)

(ii) The '*is* of the present' that means 'is *now*'. ('The sun *is* setting'.)

(iii) The 'omnitemporal *is*' that means 'is *always*'. ('Copper *is* a conductor of electricity'.)

(iv) The 'transtemporal *is*' that means 'is *throughout the present period*'. ('The earth *is* a planet of the sun'.)

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6 See REICHENBACH (1947).

7 See ŁOŚ (1941).

8 See PRIOR (1957).

9 See RESCHER (1966, item 2).

10 See VON WRIGHT (1963) and (1965). The weakness of von Wright's system relative to those of the Łoś-Prior type is demonstrated in RESCHER and GARSON (1967).

In contrast to the atemporal 'is' of (i) – let us write it as *is* – the uses of 'is' at issue in (ii)–(iv) may all be characterized as temporal. No doubt subtler gradations of meaning could be discovered, but the four that have been indicated show that 'is' is highly equivocal in that it can bear sharply divergent constructions from a chronological standpoint.[11]

B. *Translating Temporal to Atemporal IS*

Supposing the only 'is' at our disposal to be the atemporal one (*is*), the question arises whether we can still render statements like:

(S_1) It is (i.e., is *now*) raining in London.

In making such a translation, how are we to preserve the reference to *now*, the present moment? We cannot do this without doing it by an overt use of dates. It would not do to use:

(S_2) It *is* raining in London on July 1, 1966.

Even if it is correct that July 1, 1966 is the present date, these propositions will not be equivalent. For (S_1), unlike (S_2), does not tell us the date of the occurrence at issue; and (S_2), unlike (S_1), does not tell us what is happening in London *now*. The very best we can do to render (S_1) is:

(S_3) Its raining in London now *is* a fact.

If we adopt a theory of *facts* according to which these, unlike events, are in themselves timeless, and then adopt an *overt* means of temporal reference, we can render a statement like (S_1) by means of the atemporal '*is*' in the manner of (S_3).

Along these lines, we can use the atemporal '*is*' as a way of providing tenseless counterparts to tensed statements – deliberately emptying the verb-copula of such statements of any reference to the present. Thus to indicate rainfall in London on January 1, 2000 we *do* say 'It *will* rain in

[11] The logical theory of statements (both categorical and hypothetical) having a tensed copula was treated extensively in medieval times. The logical writings of such later schoolmen as Ockham, Albert of Saxony, and John Buridan treated chronological considerations extensively, *inter alia* drawing a temporally grounded distinction between an omnitemporal *consequentia simpliciter* (or *simplex*) and a temporal *consequentia ut nunc*. Some discussion of these matters can be found in E. A. Moody, *Truth and Consequence in Medieval Logic* (Amsterdam, 1953), pp. 53–57, 73–80, 97–100.

London on January 1, 2000' using a future tense form of 'is'. But we *could* say 'Its raining in London on January 1, 2000 *is* a fact', thus shifting to an atemporal '*is*'. A similar expedient is of course also possible with respect to the past. Instead of 'Caesar was assassinated in 44 B.C.' we could say 'Caesar's assassination in 44 B.C. *is* a fact'. Since facts, unlike things and events, can – on one plausible construction of the matter – be taken to be atemporal, such paraphrasing can always effect the shift from a tensed to an atemporal 'is'. But this is always rather a *transformation* than a *translation*: something is always lost in the process – to wit, the temporal placement of the event at issue with the respect to the time of assertion, i.e., the actual present. When a tensed copula is used, the statement asserted is itself made from within the temporal framework; when the atemporal *is* is used, the statement may well be *about* something that happens within the temporal framework, but the assertion itself does not have a stance within the temporal framework. Even when 'the same fact' is viewed, there is a crucial difference in perspective here – a difference so wide that there is no way to bridge it over.

C. *Chronologically Definite and Indefinite Statements*

We shall say that a statement is *chronologically definite* if its truth or falsity is independent of the time at which it is asserted. Consider the statements:

(1) It sometimes rains in London.
(2) It always rains in London.
(3) Its raining in London on all Sundays of 3000 A.D. is a fact.
(4) Its raining in London on January 1, 3000 A.D. is a fact.

These statements are all *chronologically definite*: Their truth or falsity is entirely unaffected, no matter what answer is given to the question 'When was that statement made (asserted)?' The assertion of time of the statement has no bearing on its truth-value.

By contrast, consider the statements:

(5) It is now raining in London.
(6) It rained in London yesterday.
(7) It will rain in London sometime next week.

These statements are all *chronologically indefinite* in that their truth or

falsity is not independent of their time of assertion. The answer given to the question 'When was that statement made?' is essential rather than immaterial to the determination of the truth or falsity of the statement.[12]

In view of this chronological indefiniteness, some may feel inclined to deny to (5)–(7) the status of 'genuine' statements, on the grounds that their truth or falsity depends in an essential way upon matters that are not explicitly contained in the verbal meaning-content of the statement itself, so that the proposition expressed by the statement is left indeterminate – to look at the matter in a somewhat old-fashioned way. On such a view of the matter, chronologically indefinite statements would be assimilated, say, to pronominally ambiguous statements such as '*His* father is tall' or '*Their* house is large'.

If someone wishes to limit the applicability of the classification-label 'statement' in this way – and consequently to coin some grouping of 'quasi-statements' to include chronologically indefinite 'statements' as well as 'proper statements' – there is nothing whatever to object. (Our entire enterprise could be carried through in this way at the cost of added complications.) But we shall not adopt this policy here, and shall speak of both of these types of declarations as 'statements', and shall use the usual statements variables '*p*', '*q*', '*r*', etc., to represent temporally indefinite as well as temporally definite statements. Our reason for the policy is that a logic of chronological propositions must be prepared to cope with both types of statements, and that uniformity of treatment leads to certain economies, while any possible confusions can be avoided by a careful heed of the necessary distinctions.

D. *Dates and Pseudo-Dates*

It is useful to distinguish between dates and pseudo-dates. A genuine, definite *date* is a time-specification that is *chronologically stable* (such as 'January 1, 3000' or 'The day of Lincoln's assassination'). A *pseudo-date* is a time specification that is *chronologically unstable* (such as 'today' or

[12] Cf. J. T. Saunders, 'A Sea Fight Tomorrow?' *The Philosophical Review* **67** (1958) 367–378 (see especially pp. 373–376). A discussion of 'Fugitive Propositions', represented by statements which, like 'It is now snowing', may be said to 'express different propositions when made on different occasions' was set afoot by a paper by A. E. Duncan-Jones in *Analysis* **10** (1949–50) 21–23. Cf. also a paper of the same title by P. Nowell Smith, *ibid.*, 100–103, as well as the illuminating paper on 'Tense Usage and Propositions' by Jonathan Cohen, *ibid.*, **11** (1951) 80–87.

'six weeks ago'). The specification of a date does not change with the particular occasion of reference whereas the specification of a pseudo-date does change with this occasion, because of the tacit reference to the transient present.

E. *Times of Assertion*

Let us introduce the notation,

$$|t \vdash p$$

to represent the assertion of p at the time t. For example if p_1 is the statement 'It is raining in London today', and t_1 is January 1, 1900, then '$|t_1 \vdash p_1$' represents the assertion made on January 1, 1900 that it is raining today – an assertion that is true if and only if the statement 'Its raining in London on January 1, 1900 *is* a fact' is true.

When we consider a complex assertion of the type

$$|t \vdash p$$

we note that there are four possibilities:

(1) p is chronologically definite and t is a (genuine) date
(2) p is chronologically definite and t is a pseudo-date
(3) p is chronologically indefinite and t is a (genuine) date
(4) p is chronologically indefinite and t is a pseudo-date.

In cases (1) and (2) when p is *chronologically definite*, then (by definition) '$|t \vdash p$' and '$|t' \vdash p$' are materially equivalent (i.e. have the same truth-value) for all values of t and t'. The assertion times – and thus the dating schemes – become irrelevant: the truth-status of the complexes at issue will hinge simply and solely upon that of 'p' itself.

Consider an instance of type (3):

(3′) |January 1, 2000 ⊢ It rains in London tomorrow.

This complex assertion is materially equivalent with (i.e., has the same truth-value as) the *chronologically definite* proposition:

(3″) It rains in London on January 2, 2000.

And this situation will, of course, prevail generally in case (3).

The state of affairs in case (4) is analogous. Consider, for example:

(4′) |Yesterday⊢It rains in London tomorrow.

This complex assertion is materially equivalent with the *chronologically indefinite* proposition:

(4″) It rains in London today.

And this situation will, of course, prevail generally in case (4).

The question of the existence of certain logico-chronological relationships among chronological propositions is a subject of some interest. For example, let us suppose time to be measured in units of days, so that the time variable is discretized. Thus let $(t+1)$ represent 'tomorrow', $(t-1)$ 'the day before t-day', and the like. Let the statements p_1, q_1, and r_1 be as follows:

p_1: It rains in London today.
q_1: It will rain in London tomorrow.
r_1: It rained in London yesterday.

Consider the following assertions:

(P) $|t \vdash p_1$
(Q) $|t-1 \vdash q_1$
(R) $|t+1 \vdash r_1$.

It is clear that, for any value of t whatsoever, the assertions (P), (Q), and (R) must (logically) be materially equivalent (i.e., have the same truth-value). This rather trivial illustration establishes the far-reaching point – to which we shall have to return below – that the theory of chronological propositions must be prepared to exhibit the existence of logical relationships among these propositions of such a kind that the truth of the assertion of one statement at one time may be bound up essentially with the truth (or falsity) of the assertion of some very different statement at another time.[13]

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[13] It should be noted that the Tarski criterion of truth

'S' is true if and only if S

cannot be applied to chronologically indefinite statements without due modification to take account of assertion times. It clearly would not do to assert

'It is raining in London now' is true if and only if it is raining in London now,

if the time of assertion of this entire statement differs from that of the quoted substatement.

F. *Two Styles of Chronology*

The distinction between dates and pseudo-dates points to the existence of two very different chronological dating-procedures, depending upon whether the fundamental reference-point – the 'origin' in mathematical terms – of the chronological scheme is a chronologically stable *date* or a chronologically unstable *pseudo-date*. If the 'origin' is a pseudo-date, say 'today', we shall have a style of dating all of whose chronological specifiers are pseudo-dates, e.g. tomorrow, day-before yesterday, four days ago, etc. If, on the other hand the 'origin' is a genuine date – or rather, a concrete, particular event fixing such a date – say *the founding of Rome*, or *the accession of Alexander*, we shall have a style of dating all of whose dates are also of this chronologically stable type, *e.g. two hundred and fifty years 'ab urbe condita'*.

The style of chronology that is adopted will of course have significant implications for the ensuing logic of chronological propositions. If the style of chronology is based upon pseudo-dates, all statements to the effect that a certain event occurs or a certain process takes place at such and such a time will be chronologically indefinite statements, whereas if the chronology is based upon genuine dates, such statements will all be chronologically definite. In this derivative sense we may thus speak of a chronology of genuine dates as being chronologically definite or stable, and one of pseudo-dates as being chronologically indefinite or unstable.

3. THE CONCEPT OF CHRONOLOGICAL REALIZATION

Let p be some chronologically indefinite statement. Then we can in general form another statement asserting that p holds (obtains) at the particular time t. Correspondingly, we introduce the statement-forming operation R, the operation of *chronological realization*. We shall write '$R_t(p)$' to be read 'p *is* realized at the time t', which is to represent the explicit statement that p holds (obtains) specifically at the time t. Thus if t_1 is 3 *p.m. Greenwich time on January* 1, 2000, and p_1 is the (chronologically indefinite) statement, 'All men are (i.e., are *now*) playing chess', then '$R_t(p_1)$' is the statement 'It is the case at 3 p.m. Greenwich time on January 1, 2000 that all men are (now) playing chess', or equivalently simply, 'All men are playing chess at 3 p.m. Greenwich time on January 1,

2000'. Again, if p_2 is the statement 'All men will play chess tomorrow', then '$R_{t_1}(p_2)$' is the statement 'It is the case at 3 p.m. Greenwich time on January 1, 2000 that all men will be playing chess tomorrow'.

We shall abstract from one possible difficulty that can arise with this schematism, namely that the time units of p and of t are incompatible so that '$R_t(p)$' would be senseless. For example if p is the statement 'It has now been raining for exactly one minute', then we can hardly say that p is the case *on a certain day* or *in a certain year*. We shall simply assume that p and t are compatible in all cases we are considering.

If t is a proper date (not a pseudo-date), then '$R_t(p)$' is always temporally definite. For example, if p_1 is the temporally indefinite statement 'It is raining in London today', and t_1 is as specified two paragraphs ago, then '$R_{t_1}(p_1)$' is the temporally definite statement, 'It is raining in London at 3 p.m. Greenwich time on January 1, 2000'. On the other hand, if t_2 is a pseudo-date 'tomorrow', then '$R_{t_2}(p_1)$' is 'It is the case tomorrow that it is raining in London today'.

In defining $R_t(p)$ we have thus far supposed that p is a chronologically indefinite statement. It will prove convenient to drop this restriction by means of the following:

Convention

> If p is a chronologically definite statement, then '$R_t(p)$' is to be taken simply as equivalent with p itself, for any value of t whatsoever (this now being arbitrary). Thus if p is chronologically definite, then 'p' and '$(\forall t)\, R_t(p)$' are to be regarded as equivalent.

In short, a chronologically definite statement is to be taken as realized omnitemporally, i.e., at all times whatsoever.

Our discussion of chronological realization has to this point glossed over one point of ambiguity. Let p_1 be 'It was raining in London yesterday' and let t_1 be *the day after tomorrow*. Then '$R_{t_1}(p_1)$' is: 'It is the case the day after tomorrow that it was raining in London yesterday'. The ambiguity at issue relates to the reference point for the final 'yesterday' of this sentence. Does it mean (i) yesterday *as of now*, or (ii) yesterday *as of then*; i.e., is the statement at issue equivalent to 'It was raining in London yesterday' or 'It will be raining in London tomorrow'? This ambiguity

can be pinpointed by asking how an iteration of the R-operation is to be construed in a chronology based on the pseudo-date 'now' as origin. That is, are we to interpret '$R_{t'}[R_t(p)]$' as

(I) It is the case t' time units from now that it is the case t time units from (*the original*) *now* that p – or equivalently simply as '$R_t(p)$',

or as

(II) It is the case t' time-units from now that it is the case t time-units thence (i.e., from the *new* 'now') that p – or equivalently as '$R_{t'+t}(p)$'.

Since each of these represents a feasible alternative policy for the construction of the relationship R, we shall have to examine the consequences of an adoption of either. (Note that – in view of the foregoing 'Convention' – these two constructions will simply coincide in the case of chronologically definite propositions.)

4. AXIOMS FOR THE LOGICAL THEORY OF CHRONOLOGICAL PROPOSITIONS

We shall now set out to develop the logical theory of chronological propositions in a systematic, formal way. The machinery we shall require for this purpose is as follows:

(i) 'p', 'q', 'r', etc., as propositional variables, ranging over both chronologically definite and indefinite statements.

(ii) 't', 't'', etc., as variables for periods of elapsed time since the 'origin', i.e. as dates or quasi-dates.

(iii) The usual connectives of propositional logic, specifically '$\sim$' negation, '&' for conjunction, '$\rightarrow$' for implication, and '$\leftrightarrow$' for mutual implication or equivalence. (We shall leave open the option of construing '$\rightarrow$' either as material implication, or as a stronger mode of implication such as strict implication or entailment.)

(iv) The usual quantifiers, both universal ($\forall$) and existential ($\exists$). These will be needed at present to range not over *individuals* – since we are developing only a propositional logic, not a functional calculus – but only over times (dates).

(v) The operator R for chronological realization, as introduced above.

We shall need, moreover, to presuppose all of the usual logical principles for the propositional connectives and for quantifiers. The only novelty is to be the set of logical rules for the operator R to whose elaboration we now turn.

We shall suppose, to begin with, that the operator R is governed by the following rules or theses:

(T1) The negation of a statement is realized at a given time *iff* (*if and only if*) it is not the case that that statement is realized at that time:

$$R_t(\sim p) \leftrightarrow \sim R_t(p).$$

(T2) A conjunction of two statements is realized at a given time *iff* each of these two statements is realized at that time:

$$R_t(p \,\&\, q) \leftrightarrow [R_t(p) \,\&\, R_t(q)].$$

If a statement is realized at any and every time whatsoever, then it is true simpliciter, without temporal qualification. We thus have:

(T3) If a statement is realized universally at all times, then it obtains unqualifiedly:

$$(\forall t)\, R_t(p) \rightarrow p.$$

For any statement p (chronologically definite or not) and any style of chronology (chronologically definite or not), '$(\forall t)\, R_t(p)$' is a chronologically definite statement, and so for that matter is '$(\exists t)\, R_t(p)$'. It thus falls under the Convention of Section 3 above, with the result that:

(T4) Being a chronologically definite proposition '$(\forall t)\, R_t(p)$' is such that a statement asserting its realization at any given time is equivalent with '$(\forall t)\, R_t(p)$' itself:

$$R_{t'}[(\forall t)\, R_t(p)] \leftrightarrow (\forall t)\, R_t(p).$$

Finally there are – as we saw in Section 3 above – two distinct ways

of construing iterations of the R-operation. We shall thus have a choice between the rules.

(T5–I) $R_{t'}[R_t(p)] \leftrightarrow R_t(p)$
(T5–II) $R_{t'}[R_t(p)] \leftrightarrow R_{t'+t}(p)$.

Taking these rules as our starting point, we shall now present an axiomatic theory – or rather two axiomatic theories – for the logic of the operation of chronological realization.

We have arrived at the following rules for a logic of chronological propositions:

(T1) $R_t(\sim p) \leftrightarrow \sim R_t(p)$
(T2) $R_t(p \,\&\, q) \leftrightarrow [R_t(p) \,\&\, R_t(q)]$
(T3) $(\forall t)\, R_t(p) \rightarrow p$
(T4) $R_{t'}[(\forall t)\, R_t(p)] \leftrightarrow (\forall t)\, R_t(p)$ [14]
(T5–I) $R_{t'}[R_t(p)] \leftrightarrow R_t(p)$
(T5–II) $R_{t'}[R_t(p)] \leftrightarrow R_{t'+t}(p)$.

The fifth thesis alone will determine whether our construction of $R_t(p)$ is to be of Type I or Type II in the sense of the concluding paragraph of Section 3 above.

The axiom-system based on theses (T1) to (T4) plus (T5–I) will be designated as System SI; and that based on theses (T1) to (T4) plus (T5–II) will be designated as System SII. In both of these systems we shall suppose also the following rule of inference:

(R) Whenever T is a thesis, so is $(\forall t)\, R_t$(T).

Notice that if we were to add to our theses the converse of (T3), interesting – but dire – consequences would ensue. For to add the thesis

$$p \rightarrow (\forall t)\, R_t(p)$$

would, in effect, be to say that all statements are temporally definite – or

[14] It will be shown in the next chapter (see especially the Appendix) that there is good reason (and little substantial loss) in changing this axiom to $R_{t'}[(\forall t)\, R_t(p)] \leftrightarrow (\forall t) R_{t'}[R_t(p)]$. In this version the principle, rather than relating to the temporal definiteness of '$(\forall t)\, R_t(p)$', becomes a thesis about universal quantification over such expressions to the effect that a prefixed realization operator $R_{t'}$ can without loss be moved across the initial universal quantifier.

simply to limit our purview to temporally definite statements, if you prefer.[15]

That the consequence of this is to abandon the chronological character of our logic, and to collapse it into ordinary, achronological logic is shown by the following line of reasoning:

Given the thesis at issue, we obtain, in view of (T3),

$$p \leftrightarrow (\forall t)\, R_t(p),$$

whence by (T4) we have it that

$$R_{t'}(p) \leftrightarrow p$$

obtains for any t' whatsoever. It is clear that *this* rule would render otiose any invocation of chronological considerations, and specifically would collapse all six of our theses (T1)–(T5–II) into explicit identities of ordinary propositional logic.

Several simplifications of these two axiom-systems can be effected as follows:

(I) (T2) can (in both systems) be replaced by one half of itself, namely,

(T2.1) $[R_t(p) \,\&\, R_t(q)] \rightarrow R_t(p \,\&\, q)$

in view of the following argument:

(1)	$R_t[(p \,\&\, q) \rightarrow p]$	by Rule (R)
(2)	$R_t(p \,\&\, q) \rightarrow R_t(p)$	from (1) by lemma (L1) [16]
(3)	$R_t(p \,\&\, q) \rightarrow R_t(q)$	by parity with (1), (2)
(4)	$R_t(p \,\&\, q) \rightarrow [R_t(p) \,\&\, R_t(q)]$	from (2), (3)

(II) Given (T1), (T5–I) can be replaced by half of itself, namely,

(T5.1–I) $R_{t'}[R_t(p)] \rightarrow R_t(p)$

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[15] Note, however, that the rule of inference corresponding to this complicated thesis,

(1) If $\vdash p$ then $\vdash (\forall t)\, R_t(p)$

is harmless, and indeed desirable, for it is simply our rule (R).

[16] The thesis in question is the lemma

(L1) $R_t(p \supset q) \rightarrow [R_t(p) \supset R_t(q)]$

whose proof is immediate once (T1) and (T2.1) are given.

in view of the derivability of its inverse by the following argument:

(1)	$R_{t'}[R_t(\sim p)] \rightarrow R_t(\sim p)$	by (T5.1–I)
(2)	$\sim R_t(\sim p) \rightarrow \sim R_{t'}[R_t(\sim p)]$	from (1)
(3)	$R_t(\sim \sim p) \rightarrow R_{t'}[R_t(\sim \sim p)]$	from (2) by (T1)
(4)	$R_t(p) \rightarrow R_{t'}[R_t(p)]$	from (3)

(III) Given (T1), (T5–II) can be replaced by one half of itself, namely,

(T5.1–II) $R_{t'}[R_t(p)] \rightarrow R_{t'+t}(p)$

in view of the derivability of its inverse by a line of argument wholly parallel with that of (II) above.

(IV) Given (T5–I), (T4) can be replaced by one half of itself, namely,

(T4.1) $(\forall t)\, R_t(p) \rightarrow R_{t'}[(\forall t)\, R_t(p)]$

in view of the derivability of its converse by the following argument:

(1)	$R_{t'}[(\forall t) R_t(p) \rightarrow R_t(p)]$	by (R) and QL (quantificational logic)
(2)	$R_{t'}[(\forall t) R_t(p)] \rightarrow R_{t'}[R_t(p)]$	from (1) by (T1)–(T2)
(3)	$A/(2) \rightarrow R_t(p)$	from (2) by (T5–I)
(4)	$A/(2) \rightarrow (\forall t) R_t(p)$	from (3) by QL.

(V) Given (T5–II), (T4) can be replaced by one half of itself, namely, as before,

(T4.1) $(\forall t)\, R_t(p) \rightarrow R_{t'}[(\forall t)\, R_t(p)]$ [18]

in view of the derivability of its converse by the following arguments:

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[17] We use '$A/(2)$' to abbreviate 'the *antecedent* of the implication-statement (2)' and '$C/(2)$' to abbreviate 'the *consequent* of the implication-statement (2)'.

[18] It is readily shown that nothing would be changed by replacing (T4.1) by:

(T4.2) $(\forall t)\, R_{t'}\, [R_t\, (p)] \rightarrow R_{t'}[(\forall t)\, R_t(p)]$.

(1)–(2) exactly as in (IV) above
(3) $A/(2) \rightarrow R_{t'+t}(p)$ from (2) by (T5–II)
(4) $A/(2) \rightarrow (\forall t)\, R_{t'+t}(p)$ from (3) by QL
(5) $(\forall t)\, R_{t'+t}(p) \rightarrow (\forall t)\, R_t(p)$ by Principle (P1) [19]
(6) $A/(2) \rightarrow (\forall t)\, R_t(p)$ from (4), (5).

We thus see that the two systems can be based on the following sets of axioms: (T1), (T2.1), (T.3), and (T4.1), for both SI and SII, and (T5.1–I) and (T5.1–II) for SI and SII respectively. The systems of chronological logic discussed in the present chapter are, in essentials, variants of two systems presented by Prior.[20] The chronological approach to modal logic was used in Greek antiquity by Aristotle, and was developed with considerable sophistication by the Megarian and Stoic logicians.[21] Their ideas on this subject were carried forward, and extended with considerable complexity over the area of syllogistic logic by the medieval Arabic logicians and by the Latin Schoolmen.[22] After the Renaissance, the whole subject lay dormant. Interest in it was revived in recent years in the wake of Professor A. N. Prior's pioneering book, *Time and Modality*.[23]

5. TEMPORAL MODALITY IN THE SYSTEMS SI AND SII

We can introduce into our systems of chronological propositions the 'Aristotelian modalities'[24] – or perhaps better 'Megarian modalities' – of *possibility* (symbolized by '*P*') and *necessity* (symbolized by '*N*') by the definitions:

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[19] The principle at issue – whose justification is at once guaranteed if one assumes that time has no beginning, i.e., that there is no unqualifiedly first date – is:

(P1) $(\forall t)\, [R_{t'+t}(p)] \leftrightarrow (\forall t)\, [R_t(p)]$.

[20] This fact is established in detail in this author's paper 'On the Logic of Chronological Propositions', *Mind* **75** (1966) 75–96.

[21] Important aspects of the history of the subject are explored in N. Rescher, 'Truth and Necessity in Temporal Perspective' in R. M. Gale (ed.), *The Concept of Time* (New York, 1967).

[22] For these medieval discussions see N. Rescher, *Temporal Modalities in Arabic Logic* (Dordrecht, 1966).

[23] Oxford, 1957.

[24] See Jaakko Hintikka, 'Necessity, Universality, and Time in Aristotle', *Ajatus* **20** (1959) 65–90.

(D1) $P(p)$ for $(\exists t)\, R_t(p)$
(D2) $N(p)$ for $(\forall t)\, R_t(p)$. [25]

As Prior has shown, when this construction of modality is introduced into his analogue to our SI, the resulting system of modal logic is identical with Lewis's system S5. Prior has also shown that the modal system generated by his analogue to our SII will be at least as strong as Lewis's system S4; but, as a matter of fact, it can be shown that the resulting system is once again S5.

The sole point of difference between S4 and S5 is that the latter system includes the additional thesis:

$$P(N(p)) \rightarrow p.$$

Since '$N(p) \rightarrow p$' obtains in general, it would serve our purpose to obtain the stronger thesis:

$$P(N(p)) \rightarrow N(p).$$

Our task, then, is to establish in SII the theorem:

$$(\exists t')R_t[(\forall t)\, R_t(p)] \rightarrow (\forall t)\, R_t(p).$$

But this is an immediate consequence of (T4).

Thus, given the definition (D1)–(D2), both of the systems SI and SII for the logic of chronological propositions lead to a system of Aristotelian modalities equivalent with Lewis's system S5.

This result is, of course, only to be expected since the sole fundamental difference between the systems SI and SII relates to the significance of R-iteration, a point of difference which clearly can have no consequences of a modal character, when the modalities are constant in the temporal manner now at issue.

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[25] It is of interest that in System SI

$$N[R_t(p)] \leftrightarrow R_t(p)$$

when N is construed by Definition (D2). For then:

(1) $N[R_t(p)] \leftrightarrow (\forall t')\, [R_{t'} R_t(p)]$
(2) $A/(1) \leftrightarrow (\forall t')\, R_t(p)$ by (T5–I)
(3) $A/(1) \leftrightarrow R_t(p)$ by quantificational logic

6. AN ALTERNATIVE CONVENTION FOR TEMPORALLY DEFINITE STATEMENTS

The reader will recall the 'Convention' regarding temporally definite statements of Section 3 above – namely that if p is chronologically definite, then '$R_t(p)$' is to be taken as equivalent with p itself. Given this principle, if p is temporally definite, then, on the Aristotelian construction of necessity of the preceding section, we have: $p \leftrightarrow N(p)$. We are led to a version of the medieval precept, *unumquodque quando est oportet esse*. Together with the Aristotelian modalities our convention would lead to classing 'Its raining in London on April 11, 1963 is a fact' as a necessary statement. This consequence suggests the desirability of exploring alternatives to the Convention of Section 3.

If p is the paradigmatically temporally definite statement 'Its being the case that ... is a fact on the (definite) date t' we can define 'p^*' as the derivative statement: 'It is *now* the case that....' We can then introduce the

Alternative Convention

> If p is a chronologically definite statement, '$R_t(p)$' is to be taken as automatically equivalent with '$R_t(p^*)$'.

This approach would, in effect, restrict the proper application of the R_t operation to temporally indefinite statements. This alternative conviction – whose further consequences we shall not now explore – would render the Aristotelian construction of modality compatible with the view that temporally definite statements need not invariably be either necessary or impossible. For the inference from '$R_t(p)$' to '$R_{t'}(p)$', which was previously automatic in the case of temporally definite statements, is no longer a feasible move. Moreover, it is obvious that for this convention the appropriate iteration-rule for R_t will have to be (T5–I), so that we are now constrained to adhere to the system SI.

7. PROCESS-IMPLICATION

Consider the (chronologically indefinite) statements:

p_1: Today is Monday.
q_1: Today is Tuesday.

Here we have it that, whenever p_1 is realized on any given date, q_1 will have to be realized the next day. That is, if our dating parameter t is based upon the discrete sequence of days, we shall have:

$$(\forall t)\,[R_t(p_1) \rightarrow R_{t+1}(q_1)].$$

This example provides the backdrop for introducing the concept of what we shall term *process-implication*, a concept formalized by the definition:

$$p\,|c| \rightarrow q \text{ for } (\forall t)\,[R_t(p) \rightarrow R_{t+c}(q)].$$

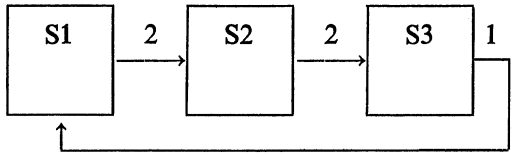
A few examples of rules that obviously hold for this relationship are as follows:

(1) $p\,|0| \rightarrow p$

(2) $(p\,|c| \rightarrow q) \rightarrow (\sim q\,|-c| \rightarrow \sim p)$

(3) $[(p\,|c| \rightarrow q)\,\&\,(q\,|d| \rightarrow r)] \rightarrow (p\,|c+d| \rightarrow r).$

This concept of implication is clearly applicable and useful in the description of processes. Suppose, for example, that we have a physical system whose behavior is characterized by the following 'transition diagram':



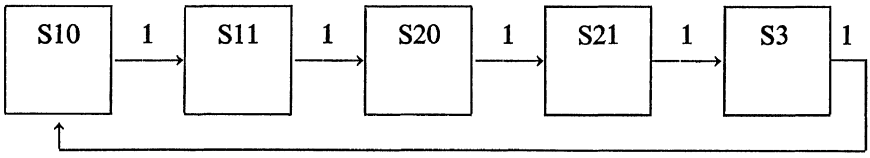
This diagram is to be interpreted as follows: When the system has been in state S1 for two time-periods, it goes into state S2; when it has been in state S2 for two time-periods, it goes into state S3; when it has been in state S3 for one time-period, it goes into state S1.

From this description of the system we obtain such process-implications as:

$$S_1\,|4| \rightarrow S_3$$
$$S_2\,|3| \rightarrow S_1$$
$$S_3\,|1| \rightarrow S_1.$$

While these implications do not characterize the behavior of the system fully, it is easy to modify the description of the system in such a way that we can characterize its behavior completely by means of process-impli-

cations. For whenever the state Si is one that repeats for n time-periods (as, e.g., S1 repeats twice in the example), we can obtain n states: Si0, Si1, ... Si($n-1$), where the system is in state Sij if it (1) is in state Si, and (2) has been in state Si for exactly j time-periods. Our system thus becomes:



The characterization of the behavior of the system is completely reproduced by the following five process-implications (each corresponding to one of the arrows of the transition diagram):

$$S10\,|1|\rightarrow S11$$
$$S11\,|1|\rightarrow S20$$
$$S20\,|1|\rightarrow S21$$
$$S21\,|1|\rightarrow S3$$
$$S3\;\;|1|\rightarrow S10.$$

Clearly, these process-implications provide exactly the same information given by the transition-diagrams that characterize the behavior of the system under discussion.

It is thus clear that the concept of process-implication affords machinery by means of which the logical theory of the concept of process can be elucidated.

8. QUANTIFICATION AND CHRONOLOGICAL REALIZATION

We have now to consider the workings of the R-operator in the presence of quantifiers – that is, quantifiers over the *individuals* of some specified domain of discourse, rather than over times (dates) as heretofore.

At the very outset, a fundamental decision will have to be made – namely whether to read such quantifiers tensedly or tenselessly, i.e., whether the existential and universal quantificational statements '$(\exists x)\ \phi x$' and '$(\forall x)\ \phi x$' are to be read as,

(I) { 'There *now exists* an x which ϕ's'.
{ 'All *now-existing* x's ϕ'.

or as,

(II) $\begin{cases} \text{'There exists (at some time or timelessly) an } x \text{ which } \phi\text{'s.} \\ \text{'All } x\text{'s that exist (at some time or timelessly) } \phi\text{'.} \end{cases}$

The difference between these constructions of quantification is of itself great, and is, moreover, critical for the introduction of R into quantificational contexts.

Let is be supposed that – instead of the 'ordinary' quantification just adverted to – we had at our disposal two styles of quantifiers: (E_) and (A_) to be existential and universal quantifiers (of the ordinary achronological type) *over times (dates) only*: and $(E_t_)$ and $(A_t_)$ to be *temporally restricted* existential and universal quantifiers *over individuals* (so that, e.g., '$(E_t x)\ \phi x$' is to mean, 'There exists an x *at the time* t such that x ϕ's (at the time t)').

The central question now at issue is whether the statements '$(\exists x)\ \phi x$' and '$(\forall x)\ \phi x$' are to be construed present-tensedly as

(I) $\begin{cases} (E_n x)\ R_n[\phi x] \\ (A_n x)\ R_n[\phi x] \end{cases}$

where n is *now*, or whether they are to be construed *timelessly* as

(II) $\begin{cases} (Et)\,(E_t x)\ R_t[\phi x] \\ (At)\,(A_t x)\ R_t[\phi x]. \end{cases}$

Only the second alternative, but not the first, renders '$(\forall x)\ \phi x$' and '$(\exists x)\ \phi x$' as temporally definite statements. This leads to the following tabulation:

Statement	Construction I	Construction II
$R_t[(\exists x)\ \phi x]$	$(E_t x)\ R_t[\phi x]$	$(Et)\,(E_t x)\ R_t[\phi x]$ [26]
$R_t[(\forall x)\ \phi x]$	$(A_t x)\ R_t[\phi x]$	$(At)\,(A_t x)\ R_t[\phi x]$
$(\exists x)\,(R_t[\phi x])$	$(E_n x)\ R_t[\phi x]$	$(Et)\,(E_t x)\ R_t[\phi x]$
$(\forall x)\,(R_t[\phi x])$	$(A_n x)\ R_t[\phi x]$	$(At)\,(A_t x)\ R_t[\phi x]$

Note now that on Construction II we have,

(T6.1) $(\exists x)\ R_t[\phi x] \rightarrow R_t[(\exists x)\ \phi x]$

(T6.2) $R_t[(\exists x)\ \phi x] \rightarrow (\exists x)\ R_t[\phi x]$

[26] We shall not even consider now such complex variants as '$(Et)\,(Et')\,(E_t x)\,R_{t'},(\phi x)$'.

and also, derivatively

(T7.1) $R_t[(\forall x)\,\phi x] \rightarrow (\forall x)\,R_t[\phi x]$
(T7.2) $(\forall x)\,R_t[\phi x] \rightarrow R_t[(\forall x)\,\phi x]$.

Of course, all of these implication-relationships would fail to obtain in general for Construction I. The logical theory of present-tense quantifiers construed according to alternative I above is substantially more complex than that of the essentially timeless construction of alternative II.

It should, incidentally, be noted further that, if one were willing to allow universal and existential quantifiers *only* with a present tense construction, *à la* alternative I, then – even if one were willing to admit of 'neutral', tenseless quantifiers over dates – one could simply not express certain propositions. One would, for instance, be unable to render: 'All snows (including those of yesteryear) are white'. It would thus seem imperative at the very least to add future-tense and past-tense quantifiers to the present-tense ones.

In a modal logic with quantification, the thesis

(B) $P[(\exists x)\,\phi x] \rightarrow (\exists x)\,P(\phi x)$

represents what Prior calls 'the Barcan formula' after Professor Ruth Barcan Marcus who first put it forward in a paper of 1948. Given the Aristotelian construction of modality of the definition (D1) of Section 5 above, this formula becomes:

(B′) $(\exists t)\,R_t[(\exists x)\,\phi x] \rightarrow (\exists x)\,(\exists t)\,R_t[\phi x]$.

This is of course equivalent with:

(B″) $(\exists t)\,R_t[(\exists x)\,\phi x] \rightarrow (\exists t)\,(\exists x)\,R_t[\phi x]$.

The question of the acceptability of (B) is thus one that can be settled by justifying the passage from '$R_t[(\exists x)\,\phi x]$' to '$(\exists x)\,R_t[\phi x]$'. But this comes down to the acceptance of thesis (T6.2). It is thus clear that the acceptability of (B) will – given the Aristotelian construction of modality – hinge upon construing quantifiers in accordance with a tenseless rather than a present-tense reading. We can certainly continue to accept (B), even with the Aristotelian construction of modality, if we are willing – as it would seem

that we ought – to construe quantifiers over individuals in the usual tenseless way.[27]

Let us introduce the symbol 'E!' with the convention that 'E!x' is to say 'x exists'. Suppose our universe of discourse to consist of individuals each of which has two real numbers associated with it (possibly $-\infty$ or $+\infty$), the first less than the second, the former representing the inauguration-date of the individual's existence, the latter its termination-date. If a is such an individual, 'R_t(E!a)' will of course be true if and only if t is a date during a's 'lifespan'. Let us return for the moment to the chronological quantifiers $(E_t\text{-})$ and $(A_t\text{-})$. Given E!, we could attempt to define these quantifiers in terms of ordinary quantification by means of the operation R, as follows:

(D1) $(E_t x)\ \phi x$ for $(\exists x)\ R_t[\text{E!}x \,\&\, \phi x]$
(D2) $(A_t x)\ \phi x$ for $(\forall x)\ (R_t[\text{E!}x] \rightarrow R_t[\phi x])$.

However this procedure is, of course, open to us only if we construe the $\exists$ and $\forall$ quantifiers as representing 'ordinary' (atemporal) quantification, and not if we construe them present-tensedly.

9. THE ELABORATION OF ARISTOTELIAN TEMPORAL MODALITY

It might be expected that the Aristotelian construction of modality would lead to certain elaborations of the chronological approach. For if one is minded to identify the *necessary* with that which is *actual all of the time*, then one might well be led by an examination of concrete cases to more sophisticated distinctions in the application of this idea. Consider the following statements:

(1) Necessarily: All men animate (i.e., exhibit animality).
(2) Necessarily: All men breathe.
(3) Necessarily: All men die.

Clearly temporal considerations of a different sort apply here, as follows:

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[27] In construing quantifiers tensedly one is forced to heap complication upon complication to an extent that casts doubt on the entire enterprise itself – particularly so since nothing can in any event be done without resorting to tenseless quantifiers (viz., over dates).

(1a) At all times, all men living (existing) at the time animate *at all times* during their life-span.

(1b) All the men living (existing) at the time breathe *at most times* during their life-span.

(1c) All the men living (existing) at the time die *at some time* during their life-span.

Plainly, then, different modes of temporal necessity are at issue in the three initial statements.

Accepting the Aristotelian temporal construction of necessity, it could yet be maintained that quite distinct senses of *necessity* are left open. For 'Necessarily: All X's ϕ', construed as 'All X's always ϕ', could be taken in any of the following ways:

(i) $(At)\,(A_t x)\,[Xx \rightarrow (At')\,(R_{t'}[\mathrm{E}!x] \rightarrow R_{t'}[\phi x])]$

(ii) $(At)\,(A_t x)\,(Xx \rightarrow R_t[\phi x])$

(iii) $(At)\,(A_t x)\,(Xx \rightarrow (Et')\,R_{t'}[\phi x])$

(iv) $(At)\,(A_t x)\,(Xx \rightarrow \phi x)$.

Here (i) and (iii) correspond to (1a) and (1c) above.[28]

We might thus expect the temporal construction of modality to eventuate in the introduction of such different chronological modes of necessity, and it is interesting to find that this expectation is met. For the famous Arabic philosopher-physician Avicenna (980–1037) discussed modal distinctions of just this kind.[29] There is some reason to believe that such elaborations go back at least to al-Fārābī (870–950) in the Arabic tradition, and very possibly go back beyond him to part of the tradition of the Greek peripatetics.[30] But in any event, it became a part of the Arabic

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[28] Alternative (1b) can be accommodated only by introducing special quantificational machinery to handle the idea of 'at most times'. See N. Rescher, 'Plurality-Quantification and Quasi-Categorical Propositions', *The Journal of Symbolic Logic* **27** (1962) 373–374, and also Section 5 of Chap. X above.

[29] His view is reported and criticized in a *Quaesitum* of Averroes, '*Quid sit propositio absoluta, id est de inesse*', printed in vol. I of *Aristotelis Opera cum Averrois Comentariis*, Venice, 1562 (photo-reprinted in Frankfurt am Main, 1962), pp. 78 *recto*–80 *recto*. For a translation of the original Arabic (the Latin being translated from a Hebrew version) see N. Rescher, 'Averroes' *Quaesitum* on the Absolute (Assertoric) Proposition' in *Studies in the History of Arabic Logic* (Pittsburgh, 1963).

[30] For a fuller discussion see N. Rescher, *Temporal Modalities in Arabic Logic* (Dordrecht, 1966).

tradition of logic in the wake of Avicenna. I shall give some samples from a very influential manual, the *Risālah al-Shamsiyyah* of al-Qazwīnī al-Kātibī (1220–c. 1280):

> There are thirteen modal propositions into which it is usual to inquire... (1) The *absolute necessary* (proposition). It pronounces that the predicate is affirmed or denied of the subject of necessity as long as the essence of the subject exists (in a thing), as when we say, 'every man is of necessity an animal'... (5) The *general absolute* (proposition). It pronounces that the predicate is actually (i.e., currently) affirmed or denied of the subject, as if we say 'every man without exception is (now) breathing'... (11) The *temporal* (proposition). It pronounces that the predicate is affirmed or denied of the subject during a definite period of the existence of the subject... (12) The *spread* (proposition). It pronounces that the predicate is affirmed or denied of the subject of necessity and during an indefinite period of the existence of the subject...[31]

Although other than chronological considerations are involved in some of the thirteen modalities at issue, it is clear from the instanced cases that we are here confronted by an attempt to elaborate on the fundamental Aristotelian linkage of modality to time, based on the idea that the necessary is that which happens all of the time.[32]

10. CONCLUSION

The logical theory of chronological propositions is of special interest from the philosophical point of view because of its linkage with various problems prominent in the history of the subject, such as Aristotle's problem about the truth-status of propositions about future-contingent matters. (If it is *now* true that I will do *A* tomorrow, then how can it be that I arrive at a decision to do A later on?) We cannot now pursue these matters, but refer the interested reader to discussions elsewhere.[33] What we have tried to do here is to develop machinery for the treatment of the relevant conceptual

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[31] A. Sprenger (ed.) *A Dictionary of the Technical Terms used in the Sciences of the Musulman*, (Calcutta 1854); 'First Appendix' issued in 1862, pp. 19–20 of the English translation.

[32] For a detailed exposition see N. Rescher, *Temporal Modalities in Arabic Logic* (*op. cit.*).

[33] See N. Rescher 'Truth and Necessity in Temporal Perspective' in R. Gale (ed.), *The Philosophy of Time* (New York, 1967).

issues with the precision afforded by the instrumentalities of symbolic logic.[34]

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[34] The present chapter presents a slightly expanded version of the author's article 'On the Logic of Chronological Propositions', *Mind* **75** (1966) 75–96. Some minor improvements in the systems presented there (and here) are effected in the next chapter.

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APPENDIX ON THE LOGIC OF DETERMINATION AND DETERMINISM

Let us introduce a modality of 'determination' subject to three guiding intuitions:

(i) If a statement is realized at a given time, its realization at that time is 'determined' for all *subsequent* times: *i.e.*, all realized statements are *post-determined.*

(ii) If a statement's realization is determined at some *prior* time, then that statement will have to be realized in due course.

(iii) If a statement is necessary, its realization at any given time is 'determined' at any time whatsoever. (Thus, in particular, the realization at a given time of a necessary statement is always pre-determined.)

Let us employ the notation '$D_t[R_{t'}(p)]$' to state that 'p's realization at the time t' is determined (to be true) at the time t'. We shall suppose that this concept is governed by the rules:

(D0) $D_t[\sim R_{t'}(p)] \rightarrow \sim D_t[R_{t'}(p)]$ for all t, t'.

(D1) $R_t(p) \rightarrow D_{t'}[R_t(p)]$ whenever $t' \geqq t$.[1]

(D2) $D_t[R_{t'}(p)] \rightarrow R_{t'}(p)$ whenever $t < t'$.

(D3) $N(p) \rightarrow D_t[R_{t'}(p)]$ for all t, t'.

(D4) If 'p' is a theorem, then so is '$N(p)$'.

The converses of (D0) and (D2) are of course to be regarded as unacceptable in general. This stipulation is critically important, since otherwise D could be simply identified with the R of System SI.

If '$N(p)$' is interpreted by the 'Aristotelian construction' of necessity as '$(\forall t)\, R_t(p)$', then (D3) becomes

$$(\forall t)\, R_t(p) \rightarrow D_{t'}[R_{t''}(p)] \quad \text{for all } t', t''$$

which is a consequence of (D1), provided that time has no beginning, so that there is no first, predecessor-less date. The other axioms also all seem perfectly innocuous. (D0) and (D4) are self-explanatory. (D1) says that what is true at a certain time is determinately so thereafter. And (D2)

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[1] It is to be shown in the sequel that a modification or restriction of this axiom is necessary.

says that what is predetermined to be true must correspondingly be true.

Given this notion of determination, we can introduce a corresponding mode of *deterministic realization* by the definition:

$$\bar{R}_t(p) \text{ FOR } D_n[R_t(p)] \quad (n \text{ for } now).$$

For this $\bar{R}$ operation we will not have,

(1) $\quad \sim \bar{R}_t(p) \rightarrow \bar{R}_t(\sim p)$

since this is equivalent with

(2) $\quad \sim D_n[R_t(p)] \rightarrow D_n[R_t(\sim p)].$

Now (2) amounts to

(3) $\quad \sim D_n[R_t(p)] \rightarrow D_n[\sim R_t(p)]$

which being the converse of (D0) is not forthcoming. But by this same reasoning, we do, of course, obtain the converse of (1), namely

(4) $\quad \bar{R}_t(\sim p) \rightarrow \sim \bar{R}_t(p).$

We can also introduce the idea of *predetermination*, to be represented by the Symbol 'D^*', by the definition:

$$D^*[R_t(p)] \text{ FOR } (\exists t')\,[(t' < t) \,\&\, D_{t'}[R_t(p)]].$$

These two concepts of chronological determination D_t and of predetermination D^* are intended to capture the Aristotelian idea that statements are sometimes, but not invariably determined as definitely true (e.g., *ex post facto*). What is to be avoided is a determination that is *complete* and all-embracing, and so provides a possible basis for fatalism, since if *everything* – including all of our own choices and actions – is predetermined, there seems little point in our deliberations, since their outcome will always be 'a foregone conclusion'. Technically, this amounts to avoiding the deterministic thesis that the realization of a state of affairs is invariably such that it was predetermined. The consequence to be avoided is represented by the thesis:

(A) $\quad R_t(p) \rightarrow D^*[R_t(p)].$

It is thus an unhappy, indeed shocking, development that A. N. Prior was able to demonstrate[2] that (A) is forthcoming in both the systems S1 and

[2] Professor Prior conveyed his (unpublished) findings in correspondence.

SII provided only that one adds the very plausible further rule for the operator D_t:

(D5) $\quad D_t[R_{t'}(p) \to R_{t''}(q)] \to \{D_t[R_{t'}(p)] \to D_t[R_{t''}(p)]\}$, for all t, t', t''.

The addition of this wholly unobjectionable rule (a straightforward analogue of a standard thesis of modal logic) provides all that is needed to derive the catastrophic consequence (A) in both systems SI and SII. (It caused me more than a little chagrin to find that I had unwittingly followed Diodorus and his Stoic confrères into an all-embracing determinism.)

The argument developed by Prior goes essentially as follows:

DEMONSTRATION IN SI

Let $t' < t$

(1)	$R_t(p) \to R_{t'}[R_t(p)]$	(T5)
(2)	$R_t(p) \to D_{t'}(R_{t'}[R_t(p)])$	(D1)
(3)	$R_t(p) \to D_{t'}[R_t(p)]$	(D3), (D4), (D5)[3]
(4)	$R_t(p) \to (\exists t'')[(t' < t) \,\&\, D_{t''}[R_t(p)]]$	(3)
(A)	$R_t(p) \to D^*[R_t(p)]$	(D*)

DEMONSTRATION IN SII

Let $t'' < t' + t'' < t + t' + t''$

(1)	$R_{t''+t'+t}(p) \to R_{t''}[R_{t'+t}(p)]$	(T5*)
(2)	$R_{t''+t'+t}(p) \to D_{t''}(R_{t''}[R_{t'+t}(p)])$	(D1)
(3)	$R_{t''+t'+t}(p) \to D_{t''}[R_{t''+t'+t}(p)]$	(D5)[4]
(4)	$R_{t''+t'+t}(p) \to (\exists t''')[(t''' < t'' + t' + t) \,\&\, D_{t'''+t'+t}(p)]$	(3)
(A)	$R_{t''+t'+t}(p) \to D^*[R_{t''+t'+t}(p)]$	(D*)

Since the 'catastrophic consequence' (A) can thus be obtained in both the systems SI and SII, it is clear that some rather fundamental change – presumably in the D-axioms – will be necessary to assert this consequence.

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[3]

(a)	$N[R_{t'}[R_t(p)] \to R_t(p)]$	(T5), (D4)
(b)	$D_{t'}[R_{t'}[R_t(p)] \to R_t(p)]$	(D3)
(c)	$D_{t'}(R_{t'}[R_t(p)]) \to D_{t'}[R_t(p)]$	(D5)

[4]

(a)	$N[R_{t''}[R_{t'+t}(p)] \to R_{t''+t'+t}(p)]$	(T5*), (D4)
(b)	$D_{t''}[R_{t''}[R_{t'+t}(p)] \to R_{t''+t'+t}(p)]$	(D3)
(c)	$D_{t''}(R_{t''}[R_{t'+t}(p)]) \to D_{t''}[R_{t''+t'+t}(p)]$	(D5)

On first thought, it might seem that a weakening of (D1) by excluding the case $t' = t$ might be what is needed. However, even if we restrict (D1) in this seemingly plausible way, (A) can still be derived, as follows:

DEMONSTRATION IN SI

Let $t' < t'' < t$

(1)	$R_t(p) \rightarrow R_{t'}[R_t(p)]$	(T5)
(2)	$R_t(p) \rightarrow D_{t''}(R_{t'}[R_t(p)])$	(D1)
(3)	$R_t(p) \rightarrow D_{t''}[R_t(p)]$	(D3), (D4), (D5)[5]
(4)	$R_t(p) \rightarrow (\exists t''')[(t''' < t) \& D_{t'''}[R_t(p)]]$	(3)
(A)	$R_t(p) \rightarrow D^*[R_t(p)]$	(D*)

DEMONSTRATION IN SII

Let $t' < t''$ and $t'' < t' + t$

(1)	$R_{t'+t}(p) \rightarrow R_{t'}[R_t(p)]$	(T5*)
(2)	$R_{t'+t}(p) \rightarrow D_{t''}(R_{t'}[R_t(p)]$	(D1)
(3)	$R_{t'+t}(p) \rightarrow D_{t''}[R_{t'+t}(p)]$	(D3), (D4), (D5)[6]
(4)	$R_{t'+t}(p) \rightarrow (\exists t''')[(t''' < t' + t) \& D_{t'''}[R_{t'+t}(p)]]$	(3)
(A)	$R_{t'+t}(p) \rightarrow D^*[R_{t'+t}(p)]$	(D*)

It is therefore clear that the proposed restriction upon (D1) will not serve to avoid the unwanted deterministic thesis (A). But it is also apparent from a careful inspection of the D-axioms that they are most vulnerable at the point of (D1). In fact our own proposal for the avoidance of (A) takes the form of a modification of (D1), as follows:

(D1a) $R_t(p) \rightarrow D_{t'}[R_t(p)]$, whenever $t \leq t'$, provided that p does not take the form '$R_{t''}(p)$' for $t'' \neq n$ (n for *now*).

Given the indicated restriction – which takes the plausible step of re-

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[5]
(a) $N[R_{t'}[R_t(p)] \rightarrow R_t(p)]$ (T5), (D3)
(b) $D_{t''}[(R_{t'}[R_t(p)]) \rightarrow R_t(p)]$ (D3)
(c) $D_{t''}(R_{t'}[R_t(p)]) \rightarrow D_{t''}[R_t(p)]$ (D5)

[6]
(a) $N[R_{t'}[R_t(p)] \rightarrow R_{t'+t}(p)]$ (T5*), (D4)
(b) $D_{t''}[R_{t'}[R_t(p)] \rightarrow R_{t'+t}(p)]$ (D3)
(c) $D_{t''}(R_{t'}[R_t(p)]) \rightarrow D_{t''}[R_{t'+t}(p)]$ (D5)

stricting the basic variable of (D1) to 'chronologically absolute' propositions – the inference from steps (1) to (2) in all of the derivations we have considered becomes invalid. For in each case this step requires a substitution for 'p' in (D1) which violates the restriction introduced in (Dla).

Alternatively, we could restrict (D1) as follows:

(Dlb) $R_t(p) \rightarrow D_{t'}[R_t(p)]$, whenever $t \leq t'$, provided that p does not itself take the form '$R_{t''}(q)$' for $t'' < t$.

This (even more intuitive) restriction blocks the move from (1) to (2) in all of the SI-demonstrations; though not in the last SII-demonstration. One could thus also avert the fatalistic consequence (A) by restricting oneself to the system SI, and accepting (D1) only in the modified form (Dlb).

In any case, the discussion of this Appendix will have served the purpose of illustrating the fact that the adequate treatment of the concepts of determination and predetermination in the framework of a rigorous logical system of chronological logic is by no means trivial, but is indeed a rather subtle matter. The devising of a set of rules of chronological logic capable of establishing – in a precise and rigorous way – a judicious line between occasional and pervasive determination is by no means as simple as one might expect it to be on first thought.[7]

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[7] This Appendix is taken from the author's paper on 'Truth and Necessity in Temporal Perspective' in R. M. Gale (ed.), *The Philosophy of Time* (New York, 1967). The body of the chapter is a revised version of the author's paper 'On the Logic of Chronological Propositions', *Mind* **75** (1966) 75–96.

CHAPTER XIII

TOPOLOGICAL LOGIC

1. INTRODUCTION

The purpose of this chapter is to present a very versatile family of logical systems of *positional* or *topological* logic. These systems – obtained by generalizing the existing systems of chronological logic – are to have a very general nature, capable of reflecting the characteristics of a wide range of logical systems, including not only chronological (also *temporal* or *tense*) logic, but also what we may call locative or place logic, and even a logic of 'possible worlds'.

2. THE P-OPERATOR

Let us add to a system of standard propositional logic the parametrized operator $P\alpha$, where '$P\alpha(p)$' is to be read and understood as 'the proposition p is realized at the position α'. Here α may be any element of *a range of positions*. These may be spatial positions indicated by Cartesian coordinates, or by any positional scheme such as seat-numbers in a lecture hall. Or again, the positions at issue may be temporal, with α ranging over the integers (for days or years) or over the real numbers (for a more refined scheme of dating).

Once the nature of the parameter α has been specified (as places, times, or the like), the 'propositions' at issue must be construed as *propositional functions* of this parameter-type. Thus if the α's represent times, the p's must be temporally indefinite propositions of the type 'It rained yesterday' and if the α's represent places, the p's must be spatially indefinite propositions of the type 'It is raining here'.

3. THREE BASIC AXIOMS

Regardless of the specific interpretation given the P-operator, the following two basic axioms will obtain:

(P1) $P\alpha(\sim p) \equiv \sim P\alpha(p)$

(P2) $P\alpha(p \,\&\, q) \equiv [P\alpha(p) \,\&\, P\alpha(q)]$.

The axiom (P1) asserts that if not-p is true at some position then it is not the case that p is true at that position, and conversely. Axiom (P1) embodies a decision to construct topological logic from a two-valued point of view: the (spatially indefinite) propositions at issue are to be either true or else false at any given position – a third possibility ('inapplicable', 'neutral', 'indeterminate', or the like) is excluded. (If this condition were dropped and a third truth-value admitted, the principal connective of (P1) would have to be changed from an equivalence to an implication.)

Axiom (P2) has it that if a conjunction is true at some position then each of the conjuncts is true at that position, and conversely.

Since (P1) and (P2) guarantee distributivity of the P-operator over a set of propositional connectives which is functionally complete, the following principle of distribution results:

(PD) The P-operator distributes itself over *all* truth-functional connectives.

We would, moreover, want to stipulate the following rule of inference:

(R) If $\vdash A$, then $\vdash P\alpha(A)$

where $\vdash$ represents assertion within the system we are endeavoring to construct. According to this rule, theorems are to be realized at any and every position whatsoever. Given the principle of distributivity (PD), this rule is interdeducible with the substitution rule:

(RS) If $\vdash A \equiv B$ and $\vdash P\alpha(A)$, then $\vdash P\alpha(B)$.[1]

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[1] The proof of the interdeducibility of (R) and (RS) is as follows:

(a) *Given* (RS), (R) *is derivable*, since we may prove $P\alpha(p \supset p)$ (from the tautology $P\alpha(p) \supset P\alpha(p)$ and (PD)). Whenever we wish to prove $P\alpha(T)$, where T is a truth of the system, we first prove $(p \supset p) \equiv \mathrm{T}$ and then apply (RS) to get $P\alpha(\mathrm{T})$.

(b) *Given* (R), (RS) *is derivable*, since we have by assumption that $\vdash (A \equiv B)$ and can thus prove $P\alpha(A \equiv B)$ by use of (R). But $P\alpha(A \equiv B)$ entails $P\alpha(A) \supset P\alpha(B)$ by (PD) and propositional logic. By assumption we were able to prove $P\alpha(A)$, and so we can get $P\alpha(B)$ by detachment.

To render the machinery of our system sufficiently powerful for our purposes, we shall want to be in a position to quantify over the parameter α in $P\alpha$. For the Axioms (P1) and (P2) we shall thus suppose an initial universal quantifier with respect to α, and similarly, we shall suppose every asserted thesis to be asserted universally with respect to its (otherwise unqualified) parameters.

Given the usual machinery of quantificational logic the following further axiom seems in order:

(P3) $\quad (\forall\alpha)\,(P\beta\,[P\alpha(p)]) \equiv P\beta\,[(\forall\alpha)\,P\alpha(p)].$

This axiom is readily motivated for at any rate those cases in which the parameter-values at issue are the natural numbers. For construing universal quantification as potentially infinite conjunction, we see that the left-hand side of (P3) becomes

$$P\beta(P_0(p))\,\&\,P\beta(P_1(p))\,\&\,P\beta(P_2(p))\,\&\ldots \quad [2]$$

whereas its right-hand side becomes

$$P\beta\,[P_0(p)\,\&\,P_1(p)\,\&\,P_2(p)\,\&\ldots].$$

But these two will be equivalent in virtue of (P2).

The axiom-systems we shall consider will all be based upon the axioms (P1)–(P3) and the rule (R), together with certain additions. These additions shall have to deal with (i) the problem of the relation of P-unqualified to P-qualified formulas, and (ii) the problem of iterations of P.

4. THE RELATION OF P-UNQUALIFIED TO P-QUALIFIED FORMULAS: THE PREFERRED POSITION ξ: A FOURTH AXIOM

How, subject to the specified conception of the P-operator, is one to construe the unqualified formula p? We certainly do want to assert the thesis,

(P4) $\quad (\forall\alpha)\,P\alpha(p) \supset p$

and thus assert also its equivalent (in view of (P1)):

(1) $\quad p \supset (\exists\alpha)\,P\alpha(p),$

[2] We shall write numerical parameter-values as subscripts.

That is, what is true at all positions is true unqualifiedly, and what is true is true at some position or other. But we would certainly *not* want to have also the converse of (1) and assert the equivalence:

(2) $\quad (\forall \alpha) P\alpha(p) \equiv p.$

For this would at once render otiose the introduction of the *P*-operator, since the assertion of a formula would then simply become equivalent to its assertion at all positions. Performing the substitution of simply p for $P\alpha(p)$ within the axioms introduced so far would reduce them all to tautologies of propositional logic.

We may wish to introduce a 'preferred position' ξ within our parameter range – like 'now' for time, or 'here' for space, or 'at the origin' for some coordinate-scheme – and then to identify the unqualified assertion of p with its assertion at this preferred position (to be represented by the constant ξ). This policy is enshrined in the axiom:

(P4′) $\quad p \equiv P\xi(p).$

Our initial thesis (P4) is, of course, an immediate consequence of this axiom (P4′). Thus we may envisage two systems of topological logic, depending on whether we choose to adopt (P4) or to take the added step of adopting (P4′). The system consisting of (P1), (P2), (P3), and (P4) will be called P; and that consisting of (P1), (P2), (P3), and (P4′) will be called P′.

5. THE ITERATION OF P: A FIFTH AXIOM: THE TWO SYSTEMS PI AND PII

How is one to construe

$$P\beta(P\alpha(p))$$

and similar formulas in which the *P*-operator occurs in a nested manner? One possible policy is to let the innermost operator be the complete determinant:

(P5.1) $\quad P\beta[P\alpha(P)] \equiv P\alpha(p).$

This amounts to assuming a fixed-point coordinate scheme with a specified origin: If p is true at the fixed position α then it is true everywhere that p is true at α.

The system based on (P1) to (P4) plus (P5.1) will be designated as PI. This system is thus to be based on the rule: (R) – If $\vdash A$, then $\vdash P\alpha(A)$ – and the following axioms:

(P1) $\quad P\alpha(\sim p) \equiv \sim P\alpha(p)$

(P2) $\quad P\alpha(p \,\&\, q) \equiv [P\alpha(p) \,\&\, P\alpha(q)]$

(P3) $\quad (\forall\alpha)\, P\beta[P\alpha(p)] \equiv P\beta[(\forall\alpha)\, P\alpha(p)]$

(P4) $\quad (\forall\alpha)\, P\alpha(p) \supset p$

(P5.1) $\quad P\beta[P\alpha(p)] \equiv P\alpha(p).$

The system which we will designate as P'I is to be identical with PI, except for replacing (P4) with the stronger axiom (P4'): $P\xi\,(p) \equiv p$. It should be noted that in both systems the presence of (P5.1) allows us to prove:

(P3*) $\quad (\forall\alpha)\, P\alpha(p) \equiv P\beta[(\forall\alpha)\, P\alpha(p)].$

Thus a positional prefix can be suppressed before *positionally definite* assertions – those in which the positional parameter does not have a free occurrence.

However in system P'I the whole positional machinery is readily seen to become superfluous in view of the result:

$$P\alpha(p) \equiv P\beta(p) \equiv p$$

The addition of (P4') makes sense only in a variant (PII-style) system.

A second possible policy is to assume a floating-point coordinate scheme with a shifting reference point: '$P\alpha(p)$' says that p is true at a position α units from 'here' (and not a fixed origin!), and '$P\beta(P\alpha(p))$' says that p is true at a position β units from *there*. Here we would assure a type of vector addition for parameter values and stipulate the axiom:

(P5.2$\oplus$) $\quad P\beta[P\alpha(p)] \equiv P(\beta \oplus \alpha)(p).$

We may designate as PII$\oplus$ the system built by adding (P5.2$\oplus$) to P, and by P'II$\oplus$ the system constructed from P' by adding (P5.2$\oplus$). Interestingly, the two systems PII$\oplus$ and P'II$\oplus$ are equivalent so long as we assume that there is a zero element (θ) such that $(\forall\alpha)\,[(\alpha \oplus \theta) = \alpha]$.[3] Thus,

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[3] Given the weaker thesis

(P4) $\quad (\forall\alpha)\, P\alpha\,(p) \supset p$

we may now prove the stronger:

making this assumption, our choice between (P4) and (P4′) becomes irrelevant and we may consequently omit the prime in the designation of topological systems which contain (P5.2 $\oplus$).

Exactly what further assumptions concerning $\oplus$ should be made will depend upon the parameter range. If the parameters α, β, γ (etc.) range over sets of Cartesian coordinates one can define:

$$\langle x_1, y_1, z_1, \rangle \oplus \langle x_2, y_2, z_2, \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle.$$

More complicated forms of 'addition' could also be introduced, varying with the structure of the space envisaged or the character of the coordinates that are used.

For instance, we may wish to develop a system of topological logic which captures the notion that space is closed and curves around on itself. Thus by traveling far enough in a straight line, one expects to return to the point of departure. We can easily specify an 'addition function' which would reflect this hypothesis about the nature of space. For simplicity's sake, let us consider a one-dimensional example in which we distinguish 10 spatial intervals: 0, (the origin-interval), and $1, \ldots, 9$. We may represent this one-dimensional spatial structure in two dimensions as in the figure opposite.

To calculate our position after travelling y intervals from a point which is x intervals from the origin, we simply add x and y, and consider the last digit of the result.

More complex functions could obviously be invented to reflect a large number of spatial structures. In fact we might envisage the development of spatio-temporal logics adequate to modern cosmological views about the structure of the four-dimensional spatio-temporal continuum. Here, the P-operator has as parameters the coordinates x, y, z, and t so that

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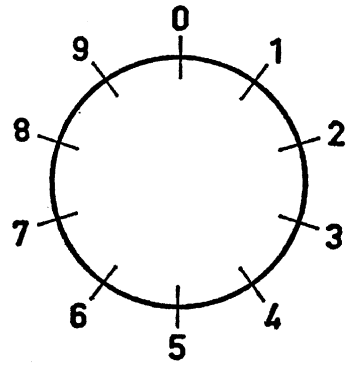
(P4′) $P\xi\,(p) \equiv p.$

Half of the proof goes thus:

$(\forall\alpha)\,(P\alpha\,(p) \supset P\alpha\,(p))$
$(\forall\alpha)\,(P\alpha P\theta\,(p) \supset P\alpha\,(p))$
$(\forall\alpha)\; P\alpha\,[P\theta\,(p) \supset p]$
$P\theta\,(p) \supset p.$

The reverse half goes thus:

$(\forall\alpha)\,(P\alpha\,(p) \supset P\alpha\,(p))$
$(\forall\alpha)\,(P\alpha\,(p) \supset P\alpha P\theta\,(p))$
$(\forall\alpha)\; P\alpha\,[p \supset P\theta\,(p)]$
$p \supset P\theta\,(p).$



'$P_{x,y,z,t}(p)$' is to be read as 'p is true at coordinates x, y, z, and time t'. The addition function appearing in (P5.2⊕) might now be a very complicated function reflecting the nature of the curvature of space-time.

The special case in which the parameters range over the real numbers is a particularly important one. Here arithmetical addition can suffice for ⊕. This brings us to the axiom-system we shall characterize as PII, unqualifiedly: This system is based on the rule (R) – If $\vdash A$, then $\vdash P\alpha(A)$ – and the following axioms:

(P1) $P\alpha(\sim p) \equiv \sim P\alpha(p)$

(P2) $P\alpha(p \,\&\, q) \equiv [P\alpha(p) \,\&\, P\alpha(q)]$

(P3) $(\forall\alpha)\,(P\beta[P\alpha(p)]) \equiv P\beta[(\forall\alpha)\,P\alpha(p)]$

(P4) $(\forall\alpha)\,P\alpha(p) \supset p$

(P5.2) $P\beta[P\alpha(p)] \equiv P(\beta+\alpha)\,(p)$.

Again, note that in the presence of (P5.2), (P3) amounts to:

$$(\forall\alpha)\,P(\beta+\alpha)\,(p) \equiv P\beta[(\forall\alpha)\,P\alpha(p)]$$

But since the function $f(x)=x+c$ maps the real-number axis into itself, $(\forall\alpha)\,(P(\beta+\alpha)(p))$ is equivalent to $\forall\alpha(P\alpha(p))$, which yields the equivalence of (P3) with:

$$(\forall\alpha)\,P\alpha(p) \equiv P\beta[(\forall\alpha)\,P\alpha(p)]$$

We again have it that positionally definite assertions can be asserted unqualifiedly, regardless of a positional prefix.

When the basic space of positions is finite or denumerable, we can associate with every (spatially indefinite) proposition its 'truth-vector',

viz., the (finite or infinite) series of T's and F's indicating its truth-status for each of the positions at issue. (If there were [say] three basic truth-values [T, F, I], the situation would be analogous, but more complex.) Regarding each such truth-vector as itself a 'truth-value' we obtain a many-valued system whose truth-tables are satisfied by the topological system PI in the usual sense that every thesis of the topological system is a tautology of the many-valued system. (A formula is a many-valued tautology when it assumes the [uniquely] 'designated' truth-values, viz., that composed uniformly of T's, for every assignment whatsoever of truth-values to its propositional variables.) This point of view also provides the basis for a semantical interpretation of the systems at issue. See Appendix II for a consideration of developments along these lines.

6. CHRONOLOGICAL LOGIC

We shall now briefly develop the logical theory of chronological propositions in a systematic, formal way. We shall suppose, to begin with, an operator R_t for chronological realization with '$R_t(p)$' to mean 'p is realized at the time t', governed by the following rules or theses:

(T1) The negation of a statement is realized at a given time *iff* (*if and only if*) it is not the case that that statement is realized at that time:
$R_t(\sim p) \equiv \sim R_t(p)$.

(T2) A conjunction of two statements is realized at a given time *iff* each of these two statements is realized at that time:
$R_t(p \,\&\, q) \equiv [R_t(p) \,\&\, R_t(q)]$.

If a statement is realized at any and every time whatsoever, then it is true simpliciter, without temporal qualification. We thus have:

(T3) If a statement is realized universally at all times, then it obtains unqualifiedly:
$(\forall t)\, R_t(p) \supset p$.

The R_t-operator can be moved past an irrelevant quantifier, so that $R_{t'}[(\forall t)\, R_t(p)]$ is to be equivalent with $(\forall t)\,(R_{t'}[R_t(p)])$ and thus correspondingly with the existential quantifier:

(T4) $R_{t'}[(\exists t)\, R_t(p)] \equiv (\exists t)\,(R_{t'}[R_t(p)])$.

Finally there are two distinct ways of construing iterations of the R_t-operator. We shall thus have a choice between the rules:

(T5-I) $R_{t'}[R_t(p)] \equiv R_t(p)$
(T5-II) $R_{t'}[R_t(p)] \equiv R_{(t'+t)}(p)$.

Taking these rules as our starting point, we arrive at an axiomatic theory – or rather two axiomatic theories – for the logic of the operation of chronological realization. The axioms for these two systems, R_1 and R_2 are set out in the following tabulation. This is done there in such a way that their relationship to the predecessor systems of Łoś and Prior becomes transparent.

Axioms for Chronological Logic

The principal axiom systems for chronological logic will be outlined here in brief schematic form. The reader is referred to Chap. XII above for further details, and for a full bibliography.

I. Łoś (1948). System L.

Axioms (L1) $R_t(\sim p) \equiv \sim R_t(p)$
(L2) $R_t(p \supset q) \supset [R_t(p) \supset R_t(q)]$
(L3) $(\forall t)\, R_t(p) \supset p$

Rule (R) If $\vdash T$, then $\vdash R_t(T)$. (This rule remains applicable in II-V below.)

NOTE: Łoś also has several further axioms to govern the range of the parameter t. They are rendered superfluous by our assumption that t ranges over the real numbers.

II. PRIOR (1957). System P_1.

Axioms (A1)–(A3) = (L1)–(L3)
(A4) $R_{t'}[R_t(p)] \supset R_t(p)$.

III. PRIOR (1957). System P_2.

Axioms (A1)–(A2) = (L1)–(L2)
(A3.2) $R_n(p) \supset p$ (where n=*now* for which the numerical index 0 is to be used.)
(A4.2) $R_{t'}[R_t(p)] \supset R_{(t'+t)}(p)$
(A5) $R_{t'}[(\exists t)\, R_t(p)] \supset (\exists t)\, R_{t'}[R_t(p)]$

IV. RESCHER (1966). System R_1.

(R1) = (L1)
(R2) $R_t(p \& q) \equiv [R_t(p) \& R_t(q)]$
(R3) = (L3)
(R4) $R_{t'}[R_t(p)] \equiv R_t(p)$
(R5) $R_{t'}[(\exists t)\, R_t(p)] \equiv (\exists t)\, (R_{t'}[R_t(p)])$.

V. RESCHER (1966). System R_2. (See Appendix I.)

(R1)–(R3)
(R4.2) $R_{t'}[R_t(p)] \equiv R_{(t'+t)}(p)$
(R5).

Apart from strictly technical results establishing formal relationships of the various systems of chronological logic to one another, the most interesting findings regarding these systems relate to their application to the theory of temporal modalities. The most striking finding here concerns the logical structure of the system of modalities, be it Megarian or Stoic:

	Megarian	Stoic
Possibly p:	$(\exists t)\, R_t(p)$	$(\exists t)\, [t \geqslant n \& R_t(p)]$
Necessarily p:	$(\forall t)\, R_t(p)$	$(\forall t)\, [t \geqslant n \supset R_t(p)]$

Prior and Rescher have shown that the structure of both these methods of defining temporal modalities used in conjunction with either R_1 or R_2 (provided the parameter t ranges over the real numbers) is captured by C.I. Lewis's well-known modal system S5.

7. RELATIONSHIP BETWEEN TOPOLOGICAL AND CHRONOLOGICAL LOGICS

It is readily seen that the systems of topological logic presented in this paper are closely related to the chronological logics of Łoś, Prior, and the author.

We notice first that the two systems of chronological logic due to the author, (R_1 and R_2) have PI and PII as exact counterparts, and that, apart from the more complicated range of the parameters envisaged for the P-operator, the systems are virtually identical. This merely underscores, as one might expect, that the logics of the spatial and temporal dimensions should have the same basic structural features.

The system due to Łoś is equivalent to only a portion of the system P_1 since there is no counterpart of axiom (A4) in the system L. If we delete (A4) from P_1, the resulting system (which we may call P_0) is equivalent to Łoś' system. Since Prior's P_1 consists of the System L plus a chronological version of the topological axiom (P5.1), P_1 is equivalent to PI with (P3) deleted, a system we may designate as P_0I.[4]

Prior's system P_2 is equivalent to PII. The two systems are alike in every respect save that the main connective of Prior's (A4.2) and (A5) is material implication '$\supset$' while the corresponding axioms of PII assert material equivalences. However, the converses of (A4.2) and (A5) can be proven in P_2; thus P_2 is a (more economical) formulation of PII.[5]

The system PII may also be related to von Wright's T-calculus. It has been shown that, *given the restriction that the temporal parameters range over the natural numbers*, a suitable translation from one system to the other can be given with the result that a modified version of R_2 and von Wright's systems are equivalent.[6]

The comparison of topological with chronological logics suggest alternative formulations for chronological logic. Specifically, we may wish to develop an analogue to one of the primed systems (P′ or P′I). In such a system we would replace (L3): $(\forall t)\, R_t(p) \supset p$ with the substantially stronger $R_0(p) \equiv p$, and thus assume that all untemporalized propositions are asserted at the 'present' time. We note that whereas Prior does do just this

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[4] The converse of (A4) is proven as follows in P_1

1.	$R_{t'}\, R_t(\sim p) \supset R_t(\sim p)$	(A4)
2.	$\sim R_{t'}\, R_t(p) \supset \sim R_t(p)$	3 applications of (A1)
3.	$R_t(p) \supset R_{t'}\, R_t(p)$.	contraposition

[5] The converse of (A4.2) and (A5) are proven in P_2 as follows:

(a) converse of (A4.2):

1.	$R_t R_{t'}(\sim p) \supset R_{t+t'}(\sim p)$	(A4.2)
2.	$\sim R_t R_{t'}(p) \supset \sim R_{t+t'}(p)$	3 applications of (A1)
3.	$R_{t+t'}(p) \supset R_t R_{t'}(p)$.	contraposition

(b) converse of A5:

1.	$R_t(p) \supset (\exists t)\, R_t(p)$	(QL)
2.	$R_{t'}\,[R_t(p) \supset (\exists t)\, R_t(p)]$	(R)
3.	$R_{t'} R_t(p) \supset [R_{t'}\, (\exists t)\, R_t(p)]$	(A2)
4.	$(\exists t)\,[R_{t'} R_t(p)] \supset R_{t'}\,[(\exists t)\, R_t(p)]$.	(QL)

[6] See N. Rescher and J. Garson, 'A Note on Chronological Logic', *Theoria* 33 (1967) 39–44. The system which is shown equivalent to von Wright's T-calculus is the chronological analogue of a topological system consisting of (P1), (P2), (P4), and (P5.2); a system which may be designated as PoII, i.e., PII minus axiom (P3).

by including (A3.2) in his system P_2, this choice between the alternatives is irrelevant in presence of (A4.2) once we assume $(\forall t)\ (t+n=t)$, i.e., that $n=0$.[7]

8. THE POSSIBLE WORLDS INTERPRETATION OF TOPOLOGICAL LOGIC

Let '$P\alpha(p)$' be construed to mean 'the proposition p is true in possible world No. α'. We are here to think of the possible worlds as enumerated in a truth-table of the Wittgenstein-Carnap manner:

Possible World No.	p_0	q_0	r_0
1	+	+	+
2	+	+	−
3	+	−	+
4	+	−	−
5	−	+	+
6	−	+	−
7	−	−	+
8	−	−	−

Note that here we have (for example) $P_3(p_0 \vee \sim r_0)$, that is, $p_0 \vee \sim r_0$ obtains in possible world No. 3. The key feature of the conception of possible worlds that is relevant for our purposes is that a 'possible' world is descriptively complete in the sense that with respect to a possible world any proposition will either be true or else false. (This feature is essential for (P1).) Moreover, we shall need to postulate some way of interpreting nested P's – say by the stipulation that (by convention) only the *innermost* P is to count:

$$P\beta\,(P\alpha(p)) \equiv P\alpha(p).$$

The iterated P's are simply ignored and treated as redundant. It is now readily seen that the resulting system of 'possible-worlds logic' is in fact isomorphic with the system PI of topological logic.

Along very closely similar lines we can also obtain an 'alternative

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[7] The converse of (A3.2): $p \supset R_n(p)$ is provable for $n=0$ in P_2 and the two theses: $p=R_n(p)$ and (A4): $(\forall t)\ R_t(p) \supset p$ are interdeducible in these systems.

systems interpretation' of topological logic. For let us conceive of several alternative systems of propositions – e.g., geometric propositions to be construed alternatively in Euclidean, Riemannian, and Lobatchevskian geometry.[8] Let α range over these three alternatives (E, R, L) and let '$P\alpha(p)$' mean 'the proposition p is true in the system α'. We shall again adopt the view that P-iterations are superfluous. The system we now arrive at will be isomorphic to PI.[9]

Again, the propositions at issue could be construed as mathematical propositional functions of a numerical parameter, with p – or now rather $p(x)$ – representing some such equation or inequality as $x^2-2=1$ or $x+3>5$. Then $P\alpha(p)$ would be taken to amount to '$p(x)$ is true when α is taken as the value of x', with α, β, etc. now ranging over some domain D of numbers. On this interpretation we would at once obtain a system of the type PI.

9. MODAL LOGIC AND TOPOLOGICAL LOGIC

The system of topological logic developed so far may be supplemented with appropriate definitions and restrictions so as to capture the modal systems M, S4, and S5. We are to regard statements as true (or false) with respect to a range of possible worlds. We might then think of the range of possible worlds as analogous to a range of positions, and so symbolize the thesis that p is true of world α as '$P\alpha(p)$'. A proposition that is asserted simply and absolutely is assumed to be asserted of the real world, or at the origin or point of reference, hence we assume the equivalence between p and $P\xi(p)$. On analogy with the Megarian modalities in chronological logic, propositions true in all possible worlds (for all positions) are considered to be necessarily true, and those true in some of the possible worlds are possibly true; hence the following definitions:

(D1) $\Box(p)$ FOR $(\forall\alpha)(P\alpha(p))$

(D2) $\Diamond(p)$ FOR $(\exists\alpha)(P\alpha(p))$.

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[8] This observation is based on an idea developed by Alan Rose in his paper, 'Eight Valued Geometry' in the *Proceedings of the London Mathematical Society* **3** (1952) 30–44.

[9] We assume that each system is such that for every formula either it or its negation is true in that system. This guarantees that $\sim P(p) \supset P(\sim p)$.

The definitions (D1) and (D2) in the system P′ produce the modal structure of M in the following sense: Every theorem of M, once translated by use of (D1) and (D2), is provable in P′, and if a formula of M is not provable in M, then its translation cannot be proved in P′. In order to reflect the stronger modal systems S4 and S5, the translations of the appropriate modal axioms must be added to the system P′.[10] To get S4 from M we add:

(1) $\Box p \supset \Box\Box p$

Thus, to reflect an S4 modal structure in a topological logic, we merely add the translation of (1):

(2) $(\forall\alpha)\, P\alpha(p) \supset (\forall\beta)\, P\beta\, [(\forall\alpha)\, P\alpha(p)]$.

This is equivalent to:

(3) $(\forall\beta)\, [(\forall\alpha)\, P\alpha(p) \supset P\beta\, ((\forall\alpha)\, P\alpha(p))]$.

Since we may quantify over free variables of theorems, we may thus write the needed addition to P′ in the following simpler form:

(P5.S4) $(\forall\alpha)\, P\alpha(p) \supset P\beta[(\forall\alpha)\, P\alpha(p)]$.

To get S5 from M we must add '$\Diamond\Box p \supset \Box p$', and consequently we must add the following axiom to P′, using the same reasoning as above:

(P5.S5) $P\beta[(\forall\alpha)\, P\alpha(p)] \supset (\forall\alpha)\, P\alpha(p)$.

However, this is not the only approach which may be used to capture the stronger modalities in a system of topological logic. If we add (P4′) to P′ thus getting the system P′I, we may easily prove both (P5.S4) and (P5.S5), and so the system P′I produces an S5 modal structure in the presence of the definitions (D1) and (D2). In the system P′II (consisting of P' plus (P5.2$\oplus$)) we can prove neither (P5.S4) nor (P5.S5) if we are allowed to make no assumptions about the properties of the operator $\oplus$ which figures in (P5.2$\oplus$).[11] Thus, *apart from any such stipulations*, the modal

[10] We wish to thank Professor Nuel D. Belnap, Jr. for assistance in establishing this fact. (For the modal systems now at issue see p. 285 below.)

[11] It could even be assumed that $x \oplus y$ is not defined everywhere; otherwise the special stipulation laid down in connection with S4, viz., $(\forall x)\,(\forall y)\,(\exists z)\,(x \oplus y = z)$, would become superfluous.

structure of M is produced. We may capture stronger modalities in the topological system P'II by making suitable assumptions about the behavior of the $\oplus$ operations, as follows:

For S4, we assume:

$$(\forall\alpha)(\forall\beta)(\exists\gamma)(\alpha\oplus\beta=\gamma).$$

From which follows: $(\forall\alpha)(\forall\beta)(\exists\gamma)[P(\alpha\oplus\beta)(p)\supset P\gamma(p)]$.
Which is equivalent to: $(\exists\alpha)(\exists\beta)[P(\alpha\oplus\beta)(p)]\supset(\exists\gamma)(P\gamma(p))$.
Which is equivalent to: $(\exists\alpha)[P\alpha(\exists\beta)[P\beta(p)]]\supset(\exists\gamma)[P\gamma(p)]$.
And so to the translation of: $\Diamond\Diamond p\supset\Diamond p$.

For S5, we assume:

$$(\forall\alpha)(\forall\beta)(\exists\gamma)(\alpha\oplus\gamma=\beta).$$

From which follows: $(\forall\alpha)(\forall\beta)(\exists\gamma)[P(\alpha\oplus\gamma)(p)\supset P\beta(p)]$.
Which is equivalent to: $(\forall\alpha)(\forall\beta)[(\forall\gamma)P(\alpha\oplus\gamma)(p)\supset P\beta(p)]$.
Which is equivalent to: $(\exists\alpha)(\forall\gamma)[P(\alpha\oplus\gamma)(p)]\supset(\forall\beta)P\beta(p)$.
Which is equivalent to: $(\exists\alpha)P\alpha(\forall\gamma)P\gamma(p)\supset(\forall\beta)P\beta(p)$.
And so to the translation of: $\Diamond\Box(p)\supset\Box(p)$.

The stipulations required to reflect S4 and S5 in the topological system P'II have interesting interpretations. When the $\oplus$ operator is construed as 'normal' arithmetical addition, (or as a pairwise addition for two-place coordinates, etc.), then the truth or falsity of the stipulations for S4 and S5 depends on the set of numbers over which the parameters α, (etc.) are thought to range. For instance, if the values of $\alpha, \beta, \gamma, \ldots$ are allowed to range only over the positive numbers, clearly the assumption required for S5: $(\forall\alpha)(\forall\beta)(\exists\gamma)(\alpha+\gamma=\beta)$ is not tenable, as no positive integer can satisfy the condition $\alpha+\gamma=\beta$ when α is greater than β. Similarly, if the values of the variables range over a continuous set of positive numbers in which there is a greatest number, the property which characterizes S4: $(\forall\alpha)(\forall\beta)(\exists\gamma)(\alpha+\beta=\gamma)$ cannot be satisfied. Clearly if the parameters range over all the reals, then an S5 modal structure results.

10. CONCLUSION

We have presented a family of systems of topological logic and have explored their relationship to various more familiar systems of logic, viz., chronological logic, modal logic, and the logic of possible worlds. No doubt much further work must be done to throw light on the nature of

such systems and to explore the range of their application. However, enough has perhaps been said here to point up the intrinsic interest of such systems and to indicate their versatility as a ground for unifying a wide variety of diverse systems of applied logic.[12]

[12] This chapter is an almost unrevised version of an article of the same title written collaboratively with James Garson and published in *The Journal of Symbolic Logic* **33** (1968).

A NOTE ON R_2

The system R_2 will, strictly speaking, be equivalent to the corresponding system of the author's 1965 paper (there called SII) only if his axiom

(T4) $\quad R_{t'}(\forall t)\, R_t(p) \equiv (\forall t)\, R_t(p)$

is changed to

(T4.2) $\quad R_{t'}(\forall t)\, R_t(p) \equiv (\forall t)\, R_{t'}R_t(p).$

For these two formulations are equivalent only for time series which have no beginning. My remark to the effect that 'nothing would be changed' by replacing (T4) with (T4.2) is not entirely correct, since once we construe the quantifier as 'long conjunction', (T4) becomes

$$[R_{t'+0}(p) \,\&\, R_{t'+1}(p)\, R_{t'+2}(p) \,\&\, \ldots]$$
$$\equiv [R_0(p) \,\&\, R_1(p) \,\&\, R_2(p) \,\&\, \ldots]$$

which is not in general the case. On the other hand, (T4.2) becomes

$$[R_{t'+0}(p) \,\&\, R_{t'+1}(p) \,\&\, R_{t'+2}(p) \,\&\, \ldots]$$
$$\equiv R_{t'}[R_0(p) \,\&\, R_1(p) \,\&\, R_2(p) \,\&\, \ldots]$$

which is, as we have seen, a *virtual* consequence of:

$$R_t(p \,\&\, q) \equiv R_t(p) \,\&\, R_t(q).$$

APPENDIX II

MANY-VALUED STRUCTURES WITHIN TOPOLOGICAL LOGIC

As a starting-point we shall postulate a basically two-valued system of topological logic, of such a sort that for any proposition p and for any value of α within the domain D, $P\alpha(p)$ will take on either the truth-value T or else the truth-value F: $/P\alpha(p)/=T$ or $/P\alpha(p)/=F$. We now use the machinery of the topological system to introduce derivative, complex truth-values as follows. The truth-value of a proposition will be *a certain subset of* D, viz., the set of all those parameter-values α for which $/P\alpha(p)/=T$:

$$//p//=\{\alpha|/P\alpha(p)/=T\}$$

Note immediately that, quite in general, the following truth-rules will be operative:

$//\neg p//=//p//'$ where S' is the complement of the set S in the domain D.

$$//p \wedge q//=//p// \cap //q//$$

$$//p \veebar q//=//p// \cup //q//$$

The many-valued system at issue will thus be isomorphic to a Boolean Algebra of sets.

We may suppose that the domain D includes a certain designated element ξ. Then all of those truth-values (i.e., subsets of D) will be designated that contain this designated element ξ. Tautologousness is to be defined in the usual way, viz., a proposition is tautologous if it invariably takes on a designated truth-value.

If a domain D includes only a single element, say $D=\{\xi\}$, then the many-valued system will have two truth-values:

$$1=\{\xi\}=V$$

$$0=\Lambda$$

The system that results from the specified truth-rules will then be:

p	$\neg p$
+1	0
0	1

p \ q	$p \wedge q$		$p \veebar q$	
	1	0	1	0
+1	1	0	1	1
0	0	0	1	0

Introducing implication and equivalence in the usual way

$$p \rightarrow q \text{ FOR } \neg p \veebar q$$
$$p \leftrightarrow q \text{ FOR } (p \rightarrow q) \wedge (q \rightarrow p)$$

we now obtain the classical propositional calculus $\mathbf{C}_2$.

If the domain D includes two elements, say $D = \{\xi, \eta\}$, then there will be four truth-values:

$$1 = \{\xi, \eta\} = V$$
$$2 = \{\xi\}$$
$$3 = \{\eta\}$$
$$4 = \Lambda$$

The many-valued system that results from the specified truth-values will be:

p	$\neg p$
+1	4
+2	3
3	2
4	1

p \ q	$p \wedge q$				$p \veebar q$				$p \rightarrow q$				$p \leftrightarrow q$			
	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
+1	1	2	3	4	1	1	1	1	1	2	3	4	1	2	3	4
+2	2	2	4	4	1	2	1	2	1	1	3	3	2	1	4	3
3	3	4	3	4	1	1	3	3	1	2	1	2	3	4	1	2
4	4	4	4	4	1	2	3	4	1	1	1	1	4	3	2	1

Thus we have here the system $\mathbf{C}_2 \times \mathbf{C}_2$, which has exactly the same tautologies as $\mathbf{C}_2$ (classical, two-valued propositional calculus) itself.

Whenever the basic logic is itself finitely many-valued – based, say on the truth-values 1, 2,..., n – then we can introduce the complex, n-place truth-value

$$//p// = \langle 1(p), 2(p), \ldots, n(p) \rangle$$

where

$$i(p) = \{\alpha | /P\alpha(p)/ = i\}$$

If the domain D contains a designated element ξ, then we may class the truth value $//p// = \langle 1(p), 2(p), \ldots, n(p) \rangle$ as designated whenever $\xi \in 1(p)$ (or, alternatively whenever $\xi \in 1(p) \cup 2(p) \cup \cdots \cup k(p)$ for some $k < n$).

The truth-rules will then depend upon those of the underlying many-valued logic.

Let us suppose the latter to be:

p	$\neg p$
+1	3
2	2
3	1

$p \wedge q$				$p \veebar q$		
p \ q	1	2	3	1	2	3
+1	1	2	3	1	1	1
2	2	2	3	1	2	2
3	3	3	3	1	2	3

Then if $//p// = \langle P_1, P_2, P_3 \rangle$ and $//q// = \langle Q_1, Q_2, Q_3 \rangle$ then:

$$//\neg p// = \langle P_3, P_2, P_1 \rangle$$
$$//p \wedge q// = \langle P_1 \cap Q_1, (P_1 \cap Q_2) \cup (P_2 \cap Q_1) \cup (P_2 \cap Q_2), P_3 \cup Q_3 \rangle$$
$$//p \veebar q// = \langle P_1 \cup Q_1, (P_2 \cap Q_2) \cup (P_2 \cap Q_3) \cup (P_3 \cap Q_2), P_3 \cap Q_3 \rangle$$

Thus consider again a domain $D = \{\xi\}$ with one single element. There will now be three three-place truth-values:

$$\text{I} = \langle V, \Lambda, \Lambda \rangle$$
$$\text{II} = \langle \Lambda, V, \Lambda \rangle$$
$$\text{III} = \langle \Lambda, \Lambda, V \rangle$$

The resulting 3-valued logic, as derived from the specified truth-rules, will be:

p	$\neg p$
+I	III
II	II
III	I

$p \wedge q$				$p \veebar q$		
p \ q	I	II	III	I	II	III
+I	I	II	III	I	I	I
II	II	II	III	I	II	II
III	III	III	III	I	II	III

Thus in this special case of a single-element domain the derived many-valued logic will be identical with the initial three-valued system.

If the domain is the two-element set $D = \{\xi, \eta\}$, then there will be nine truth-values:

$$\text{I} = \langle \{\xi, \eta\}, \Lambda, \Lambda \rangle$$
$$\text{II} = \langle \{\xi\}, \{\eta\}, \Lambda \rangle$$
$$\text{III} = \langle \{\xi\}, \Lambda, \{\eta\} \rangle$$
$$\text{IV} = \langle \{\eta\}, \{\xi\}, \Lambda \rangle$$
$$\text{V} = \langle \{\eta\}, \Lambda, \{\xi\} \rangle$$

$$\text{VI} = \langle \Lambda, \{\xi, \eta\}, \Lambda \rangle$$
$$\text{VII} = \langle \Lambda, \{\xi\}, \{\eta\} \rangle$$
$$\text{VIII} = \langle \Lambda, \{\eta\}, \{\xi\} \rangle$$
$$\text{IX} = \langle \Lambda, \Lambda, \{\xi, \eta\} \rangle$$

The resulting nine-valued logic can be shown to be the product of the initial 3-valued logic with itself.

In general, if the initial system is $\boldsymbol{X}$, then the indicated type of construction procedure will generate as the derived system the product logic $\boldsymbol{X} \times \boldsymbol{X} \times \cdots \times \boldsymbol{X}$ (n times) for an n-element domain D of possible positions.

CHAPTER XIV

ASSERTION LOGIC

1. BASIC IDEAS FOR THE LOGIC OF ASSERTION

The object of the 'logic of assertion' is to systematize the theory of the logical relationship between assertors and the propositions they assert. An 'assertor' need not in this context be a person, but may be a group of persons that issue assertions collectively, or a document, or an abstract system of propositions – such as an axiomatic theory of some subject-matter.

In general, assertion is an overt act, a biographical transaction: it is something an assertor *does* on certain chronological occasions. In constructing our logic of assertion we shall, however, abstract entirely from this historical or biographical aspect. It is important to distinguish between those specific propositions *overtly and explicitly* put forward on certain historical occasions and those to which he becomes *implicitly committed* in virtue of the explicit assertions he overtly makes. These later will be the 'tacit' assertions that are covertly or implicitly contained in what is overtly asserted, but of whose very content the assertor may well be unaware, failing entirely to realize his tacit commitment thereto. It is this second, implicit mode of assertion or 'commitment to assert' – in contradistinction to overt and explicit assertion – that is of fundamental interest for the construction of a '*logic* of assertion'.

In constructing our logic of assertion we shall supplement the orthodox machinery of propositional and quantificational logic with propositions of the type

$$Axp$$

to be interpretated as 'the assertor x asserts the proposition p'. The fact that the A-operator is to be construed in terms of a 'commitment to assert', i.e., as implicit rather than overt assertion, manifests itself in postulation of the rule:

If x asserts p, and p logically entails q, then x asserts q.

Moreover, a fundamental postulate of rationality will underly the proposed systematization of assertion logic. In part this is embodied in the just-stated rule which can be construed to say that the rational assertor is committed to the logical consequences of his assertions. A second significant feature of this postulation of rationality is the assumption that *no assertor commits himself to a contradiction.* This requirement is incorporated in the thesis:

$$\sim(\exists x)\, Ax(p \,\&\sim p).$$

Together with the indicated rule this thesis marks the fact that the assertion logic to be developed here will be a logic of *implicit rational assertion.* For in view of the rule, an assertor who asserts a contradiction asserts anything and everything. The consistency postulate rules out this uninteresting possibility – uninteresting because logically wholly undiscriminating.

We shall develop this theory of assertion for its intrinsic interest from a logical point of view. It does, however, have an intimate bearing upon the clarification of the nature of rationality, and is thus of interest also from the standpoint of philosophical applications.

2. THE SYSTEM $\mathbf{A}_1$ OF ASSERTION LOGIC

As starting point for the logic of assertion to be developed we supplement the usual machinery of orthodox logic (propositional and quantificational) with the assertion-operator A, to figure in expressions of the form Axp. We extend the principle of substitutivity of equivalents to encompass A-containing expressions. For the system $\mathbf{A}_1$, our first system of assertion logic, the following rule-axiom basis is stipulated:

(A1)	$(\forall x)\,(\exists p)\, Axp$	[Nonvacuousness]
(A2)	$(Axp \,\& Axq) \supset Ax(p \,\& q)$	[Conjunction]
(A3)	$\sim Ax(p \,\&\sim p)$	[Consistency]
(R)	If $p \vdash q$, then $Axp \vdash Axq$.	[Commitment]

The rationale of (R) and (A3) has already been dealt with in the preceding section. That of (A2) is pretty much self-explanatory. Axiom (A1) is simply a nonvacuousness postulate. It states that every assertor asserts

something, and should be regarded as a near-trivial exposition of what it means to be an assertor.[1]

From (A1) and (R) we can obtain the derived rule:

(R*) If $\vdash p$, then $\vdash Axp$ (or equivalently $\vdash (\forall x)\, Axp$).

Every assertor is committed to assert all the truths of logic. This again is a consequence of the 'rational commitment to assert' aspect of the concept of assertion with which we are here concerned to deal.

Moreover, we shall have the following theorems in $\mathbf{A}_1$:

($\mathbf{A}_1$:1) $Ax(p \,\&\, q) \equiv (Axp \,\&\, Axq)$

($\mathbf{A}_1$:2) $(Axp \vee Axq) \supset Ax(p \vee q)$

($\mathbf{A}_1$:3) $Ax{\sim}p \supset {\sim}Axp$

($\mathbf{A}_1$:4) $Axp \supset {\sim}Ax{\sim}p$

($\mathbf{A}_1$:5) ${\sim}Ax{\sim}(p \vee q) \equiv {\sim}Ax{\sim}p \vee {\sim}Ax{\sim}q$

($\mathbf{A}_1$:6) $Ax(p \vee q) \supset {\sim}Ax{\sim}p \vee {\sim}Ax{\sim}q$

($\mathbf{A}_1$:7) $Ax(p \supset q) \supset (Axp \supset Axq)$.

It is not difficult to show that the system $\mathbf{A}_1$ could also have been built up by adopting (R*) as the basic rule in place of (R), and dropping (A1)–(A3), replacing them by theorems ($\mathbf{A}_1$:3) and ($\mathbf{A}_1$:7). We also obtain a replaceability of equivalents principle:

($\mathrm{R}^{\equiv}$) If $\vdash p \equiv q$, then $\vdash Axp \equiv Axq$.

3. THE SYSTEMS $\mathbf{A}_2$ AND $\mathbf{A}_3$ OF ASSERTION LOGIC

The system to be designated as $\mathbf{A}_2$ is obtained by adding to $\mathbf{A}_1$ the axiom,

($\mathbf{A}_2$) ${\sim}p \supset (\exists x){\sim}Axp$ [Lincoln]

---

[1] In view of the rule (R), (A3) could be replaced by

$${\sim}(\exists x)\,(\forall p)\, Axp$$

or equivalently

$$(\forall x)\,(\exists p){\sim}Axp.$$

Introducing A^* as an abbreviation for ${\sim}A{\sim}$ (as will be done below), this can be reformulated first as

$$(\forall x)\,(\exists p)\, A^*x({\sim}p)$$

and equivalently as

$$(\forall x)\,(\exists p)\, A^*xp.$$

These revised versions of this principle make for an interesting comparison with (A1).

or equivalently

$(\mathbf{A}_2)$ $\quad (\forall x)\, Axp \supset p.$

This axiom states that *every falsehood is avoided by at least one assertor.* This may be dubbed 'Lincoln's axiom' because it says something analogous to his dictum that one cannot fool all of the people all of the time.

In $\mathbf{A}_2$ we will obviously also have the converse of the rule (R), viz.,:

$(\mathrm{R}^{\sim})$ $\quad$ If $\vdash (\forall x)\, Axp$ then $\vdash p.$

Thus in $\mathbf{A}_2$ we have:

$(\mathrm{R}^{\S})$ $\quad \vdash p \quad$ iff $\quad \vdash (\forall x)\, Axp.$

In any system as strong as $\mathbf{A}_2$, the logical truths, and the logical truths alone will be the demonstrably universal assertions. Note, however, that the *intrasystematic* counterpart of $(\mathrm{R}^{\S})$, viz.,

$$p \equiv (\forall x)\, Axp$$

would be patently unacceptable as a thesis of assertion logic – or rather would curtail the purview of such systems to include all, – but only – omniscient truth-assertors (viz., assertors of all truths).

The system to be designated as $\mathbf{A}_3$ is obtained by adding to $\mathbf{A}_1$ the axiom

$(\mathbf{A}_3)$ $\quad p \supset (\exists x)\, Axp$ $\qquad$ [Collective Omniscience]

and equivalently

$(\mathbf{A}_3)$ $\quad (\forall x) \sim Axp \supset \sim p \quad$ or $\quad \sim (\exists x)\, AxP \supset \sim p.$

This axiom states that *every truth is asserted by someone* or equivalently *what no one asserts is false.*

It is readily shown that $(\mathbf{A}_3)$ entails $(\mathbf{A}_2)$. For $(\mathbf{A}_3)$ is equivalent (replacing 'p' with '$\sim p$') with:

$$\sim p \supset (\exists x)\, Ax \sim p.$$

But this, given theorem $(\mathbf{A}_1{:}3)$, leads at once to $(\mathbf{A}_2)$. Thus system $\mathbf{A}_3$ includes $\mathbf{A}_2$.

It should be noted that the characteristic axioms of both $\mathbf{A}_2$ and $\mathbf{A}_3$ make strong assertions that *go beyond the strictly logical requisites* of a

theory of commitment to assert. Both are restrictive axioms that limit the purview of assertion logic to systems satisfying special (and, so to speak, *empirical*) conditions.

4. SPECIAL SITUATIONS

An assertor x may be characterized as *veridical* if everything that he asserts is true, so that whenever Axp, then p. If a system of assertion logic (conforming to the conditions of $\mathbf{A}_1$) encompasses a veridical assertor, then:

(V) $(\exists x)(\forall p)(Axp \supset p)$. [Veridicality]

Such a system will also have to satisfy the characteristic axiom of A_2:

($\mathbf{A}_2$) $(\forall x)\, Axp \supset p$.

The system obtained by adding (V) to $\mathbf{A}_1$ thus includes all of $\mathbf{A}_2$, although it will not include $\mathbf{A}_3$.

An assertor x may be characterized as *omniscient* if he asserts everything that is true, so that whenever p, then Axp. If a system of assertion logic (conforming to the conditions of $\mathbf{A}_1$) encompasses an omniscient assertor, then:

(O) $(\exists x)(\forall p)(p \supset Axp)$. [Omniscience]

Such a system will also have to satisfy the characteristic axiom of $\mathbf{A}_3$:

($\mathbf{A}_3$) $p \supset (\exists x)\, Axp$.

For suppose that (O) and not-($\mathbf{A}_3$). Then there is a p_1 such that:

$$p_1 \,\&\, (Ax) \sim \forall x p_1.$$

But by (O) there is an x_1 such that:

$$p_1 \supset Ax_1 p_1.$$

Therefore we must have $Ax_1 p_1$, which contradicts $(\forall x) \sim Axp_1$. The system obtained by adding (O) to $\mathbf{A}_1$ thus includes $\mathbf{A}_3$.

By ringing the changes of the quantifiers $(\forall p)/(\exists p)$ and $(\forall x)/(\exists x)$ as prefixes for the formulas '$Axp \supset p$' and '$p \supset Axp$,' the only nontrivial theses

that result are:[2]

(V!)	$(\forall x)\,(\forall p)\,(Axp \supset p)$	[Universal Veridicality]
(O!)	$(\forall x)\,(\forall p)\,(p \supset Axp)$	[Universal Omniscience]
(V)	$(\exists x)\,(\forall p)\,(Axp \supset p)$	
(O)	$(\exists x)\,(\forall p)\,(p \supset Axp)$.	

It is obvious that:

1. (V!) entails (V)
2. (O!) entails (O).

Moreover, it is readily demonstrated that:

3. (O) entails (V).

The following series of entailments is thus the upshot:

$$\begin{array}{l} (\text{O!}) \rightarrow (\text{O}) \rightarrow (\text{V}) \\ \qquad\qquad\quad \nearrow \\ \qquad (\text{V!}) \end{array}$$

Moreover, within the system $\mathbf{A}_1$ we have

4. (O) entails ($\mathbf{A}_3$)
5. (V) entails ($\mathbf{A}_2$)

we have that within this system

$$\begin{array}{l} \qquad\qquad\quad (\mathbf{A}_3) \rightarrow (\mathbf{A}_2) \\ \qquad\qquad\quad \nearrow \qquad\quad \nearrow \\ (\text{O!}) \rightarrow (\text{O}) \rightarrow (\text{V}) \\ \qquad\qquad\quad \nearrow \\ \qquad (\text{V!}) \end{array}$$

Thus even universal veridicality could be postulated without necessarily obtaining a system as strong as $\mathbf{A}_3$.

Two assertors x and y are *mutually agreed* if they assert exactly the same propositions, i.e., if:

$$Mxy \quad \text{FOR} \quad (\forall p)\,(Axp \equiv Ayp).$$

In a system in which all assertors are mutually agreed we will have the condition:

(M) $\quad (\exists x)\,Axp \supset (\forall y)\,Ayp$

---

[2] The 'nontrivial' here requires explanation. Consider for example, the formula '$(\exists p)\,(\forall x)\,(p \supset Axp)$'. Note that this will be a trivial truth in virtue of the rule (R*).

Two assertors x and y are *mutually contradictory* if for any 'contingent' proposition – i.e., any proposition neither not asserted nor denied by everyone – the one asserts it whenever the other asserts its denial, and the reverse:[3]

$$Cxy \quad \text{FOR} \quad (\forall p)\{[\sim(\forall z)\,Azp\,\&\sim(\forall z)\,Az(\sim p)]\supset[Axp\equiv \equiv Ay(\sim p)]\}.$$

Two assertors are *strongly mutually contradictory* if:

$$C!xy \quad \text{FOR} \quad (\forall p)\{[\sim(\forall z)\,Azp\,\&\sim(\forall z)\,Az(\sim p)]\supset[Axp\equiv \equiv \sim Ayp]\}.$$

If a system encompasses two strongly mutually contradictory assertors, it will have to satisfy axiom ($\mathbf{A}_3$), and so such a system will be at least as strong as $\mathbf{A}_3$. A system in which *all* assertors stand in strong mutual contradiction to one another must contain exactly two assertors[4] (and in consequence of this it is possible to establish that this assertion system must also satisfy the completeness axiom ($\mathbf{A}_5$) to be introduced below).

A group of assertors will be characterized as *sceptical* if every proposition not asserted by everyone is denied by someone, i.e., if we have

$$\text{(S)} \qquad \sim(\forall x)\,Axp\supset(\exists x)\,Ax(\sim p)$$

or equivalently (see n. 1 *supra*)

$$(\exists x)\sim Axp\supset(\exists x)\,Ax(\sim p) \quad \text{or} \quad (\exists x)\,A^*xp\supset(\exists x)\,Axp$$

and equivalently

$$(\forall x)\sim Ax(p)\supset(\forall x)\,Ax(\sim p) \quad \text{or} \quad (\forall x)\,A^*xp\supset(\forall x)\,Axp.$$

The addition of this thesis to $\mathbf{A}_2$ will yield $\mathbf{A}_3$. For suppose that not-($\mathbf{A}_3$), i.e.,:

$$p_1\,\&\sim(\exists x)\,Axp_1.$$

---

[3] The reason for the restriction to contingent propositions is this, that since every assertor asserts all logical truths (by our tacit-rational-commitment-to-assert approach), no two assertors can possibly disagree with respect to *every* proposition. In view of the rule

(R§) $\quad \vdash p$ iff $\vdash(\forall x)\,Axp$

the logical truths, and the logical truths alone are ruled out by this requirement of contingency, so far as $\mathbf{A}_2$–$\mathbf{A}_5$ are concerned.

[4] It is an interesting (open) question whether there is any comparable abstract condition that can be satisfied only in an assertion system with exactly one single assertor.

Then:

$$(\forall x) \sim Axp_1.$$

So by the thesis at issue we shall have:

$$(\forall x)\, Ax(\sim p_1).$$

And by (A_2) this will yield: $\sim p_1$. But this creates a contradiction, Q.E.D.

A proposition p is *contested* if some assertor asserts it and another its denial:

$$C(p) \quad \text{FOR} \quad (\exists x)\,(\exists y)\,[Axp \,\&\, Ay(\sim p)].$$

Thus we have:

$$C(p) \equiv [(\exists x)\, Axp \,\&\, (\exists y)\, Ay(\sim p)].$$

Some theorems regarding this concept are:

$$C(p) \equiv C(\sim p)$$
$$\sim C(p) \equiv [(\exists x)\, Axp \supset (\forall y)\, A^*yp].$$

The thesis that *No proposition is contested* has the three equivalent forms:

$$\sim(\exists p)\, C(p)$$
$$(\forall p) \sim C(p)$$
$$(\exists x)\, Axp \supset (\forall y)\, A^*yp.$$

It is clear that this is a thesis which could sensibly be asserted only in the setting of an assertion system so weak that 'Axp' and 'A^*xp' are non-equivalent.[5] For otherwise this thesis leads to:

$$(\exists y)\, Ayp \equiv (\forall x)\, Axp.$$

But to say this is to say that all the assertors are 'of one mind', so that there is no point in distinguishing between:

$$Axp, \quad (\exists x)\, Axp, \quad (\forall x)\, Axp.$$

This is exactly the situation when there is only one single assertor. (But note that even here one can and must preserve the distinction between p and Axp.)

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[5] And thus a system weaker than the system A_5 introduced below.

Two assertors x and y are *mutually contentious* if they agree in asserting only whatever *everyone* is agreed upon:

$$Xxy \quad \text{FOR} \quad (\forall p)\,[(Axp \,\&\, Ayp) \supset (\forall z)\,Azp].$$

If a group of assertors includes two contentious assertors, then:

$$(\exists x)\,(\exists y)\,(\forall p)\,[\sim(\forall z)\,Azp \supset [Axp \supset \sim Ayp]].$$

Whenever two observers are mutually contradictory, they will also be contentious.

A group of assertors will be *thoroughly contentious* if any two of its members are mutually contentious. In such a group, two assertors will agree in asserting only what every one asserts, with the result that whenever two members contradict one another with respect to some proposition p_1, then *all the others will be discretely neutral with respect to* p_1, and assert neither it nor its negation.

5. THE ITERATION PRINCIPLE AND THE SYSTEM $\mathbf{A}_4$

To this point we have not dealt with expressions involving *iterations* of the assertion operator, that is, with those of the form

$$Ay(\ldots)$$

where the expression in the parentheses is itself of the form $Ax(\ldots)$. Here we shall introduce an added principle postulating the equivalence of $Ax(Axp)$ with Axp. We shall then obtain a new system, $\mathbf{A}_4$, obtained by adding to $\mathbf{A}_3$ the axiom:

$(\mathbf{A}_4)$ $\quad Ax(Axp) \equiv Axp.$

This principle requires extensive discussion. This axiom can be split into two halves:

(A4.1) $\quad Ax(Axp) \supset Axp$ $\quad$ [Metahonesty]
(A4.2) $\quad Axp \supset Ax(Axp).$ $\quad$ [Metacandor]

Principle (A4.1) states that 'If an assertor asserts that he asserts something, then he actually asserts it'. It thus has it that an assertor's assertion about his own assertions are honest, and provide a correct basis for valid inferences about them. Principle (A4.2) states that 'If an assertor asserts something, then he also asserts that he asserts it'. It thus has it that an assertor

is always candid in avowing his assertions, or else that in asserting something he implicitly asserts that he asserts it. If overt (rather than implicit) assertion were at issue both these principles would be dubious, to say the least, but since we are dealing with implicit or tacit assertion they become more plausible. In point of fact, however, ($\mathbf{A}_4$) ought to be regarded as a technical – rather than natural – feature of the logic of assertion. It says that an assertor's assertion that he asserts something is to be treated as tantamount to his asserting that thing. (This could be viewed as a principle of economy, stipulating that reiterated assertion prefixes are redundant, and are simply to be treated as optical illusions. In this light, it might be looked upon less as a substantive principle than as an aspect of the machinery of our system.)

6. THE SYSTEM $\mathbf{A}_5$ OF COMPLETE ASSERTION LOGIC

An assertor x is 'complete' if he takes a definite stance with respect to *every* proposition, that is if for any and every proposition p he either asserts p or if he does not, then he asserts not-p:

$$(\forall p)\,[Axp \vee Ax(\sim p)].$$

The final system of assertion logic we shall consider, the system $\mathbf{A}_5$, is obtained by adding to $\mathbf{A}_1$ the axiom ($\mathbf{A}_4$) and also the new axiom

($\mathbf{A}_5$) $\quad Axp \vee Ax(\sim p)$ [Completeness]

or equivalently

($\mathbf{A}_5$) $\quad \sim Axp \supset Ax \sim p.$

This axiom states that every assertor either asserts any given proposition or asserts its negation, so that for any assertor x and any proposition p, x either asserts p or else x asserts not-p, in short that all the assertors are complete. $\mathbf{A}_5$ may be called the *complete system of assertion logic.*

It is readily shown that ($\mathbf{A}_5$) entails ($\mathbf{A}_3$). (This system is thus based on what is very much a 'special situation' in the sense of Sect. 4 above.) For suppose ($\mathbf{A}_5$) and not-($\mathbf{A}_3$), that is:

$$p_1 \,\&\sim(\exists x)\, Axp_1.$$

Then by ($\mathbf{A}_5$)

$$(\forall x)\, Ax(\sim p_1)$$

which entails $\sim p_1$ by $(\mathbf{A}_2)$, so that a contradiction results. Thus $\mathbf{A}_5$ includes $\mathbf{A}_3$, and $\mathbf{A}_1$–$\mathbf{A}_5$ represent a (linear) series of increasingly strong systems.

The system $\mathbf{A}_5$ is obviously a very strong one. It rules out the assertor who is prepared in some cases to be noncommittal and to assert neither p nor not-p. It stipulates that all the assertors at issue must be prepared to take a definite stand, *pro* or *con*, with regard to every proposition whatsoever.

The system $\mathbf{A}_5$ has the characteristic theorems (which do not obtain in $\mathbf{A}_4$ and its predecessors):

$(\mathbf{A}_5{:}1)$ $\quad Ax(\sim p) \equiv \sim Axp$

$(\mathbf{A}_5{:}2)$ $\quad Ax(p \vee q) \equiv (Axp \vee Axq)$.

Indeed these lead us to a characteristic feature of $\mathbf{A}_5$. Let $F = F(p, q, \ldots, r)$ be any propositional function whatsoever. We shall now have the theorem:

$$F(Axp, Axq, \ldots, Axr) = Ax[F(p, q, \ldots, r)].$$

The complete system of assertion logic $\mathbf{A}_5$ can be axiomatized directly by the following axiom set:

(1) $\quad Ax(\sim p) \equiv \sim Axp$

(2) $\quad Ax(p \,\&\, q) \equiv (Axp \,\&\, Axq)$

(3) $\quad Ax(Axp) \equiv Axp$

(R*) $\quad$ If $\vdash p$ then $\vdash Axp$.

It is readily shown that the system based on (1)–(3) plus (R*) is interdeducible with that based on (A1)–(A3), $(\mathbf{A}_4)$, $(\mathbf{A}_5)$ plus (R). This reaxiomatization thus represents a somewhat simpler and more intuitive presentation of the system $\mathbf{A}_5$.

7. WEAK ASSERTION

The assertor x will be said to assert the proposition p weakly – i.e., in the mode of *weak assertion* (A^*) – if x *does not assert not-p*:

$$A^*xp \quad \text{FOR} \quad \sim Ax(\sim p).$$

Since in all the $\mathbf{A}_i$ we have

$(\mathbf{A}_1{:}4)$ $\quad Axp \supset \sim Ax(\sim p)$

we shall invariably have

$$Axp \supset A^*xp$$

justifying our characterization of A^* as *weak* assertion.

In $\mathbf{A}_1$ – and thus in all the $\mathbf{A}_i$ – we shall have the theorems:

$$Ax(p \vee q) \supset (A^*xp \vee A^*xq)$$
$$AxA^*xp \supset A^*xp$$
$$Axp \supset AxA^*xp.$$

In the system $\mathbf{A}_5$, where alone the converse of $(\mathbf{A}_1{:}4)$ obtains, we have the stronger result:

$$Axp \equiv A^*xp.$$

Thus in $\mathbf{A}_5$ the distinction between strong and weak assertion collapses, though it does so in none of the weaker $\mathbf{A}_i$.

The characteristic axiom of $\mathbf{A}_2$ has the equivalent reformulation

(A_2') $\quad p \supset (\exists x)\, A^*p$

while the characteristic axiom of $\mathbf{A}_3$ has the equivalent reformulation:

(A_3') $\quad (\forall x)\, A^*xp \supset p.$

These two characteristic axioms will be exactly the reverse of one another if the assertion operator is stripped of its star.

In general, it is clear that the logic of assertion could be built up by taking A^* as a primitive, in place of A. The logical systems for this assertion concept would be axiomatized in a manner essentially dual to that for A, so that, for example, axiom (A2) would come to:

$$A^*x(p \vee q) \supset (A^*xp \vee A^*xq).$$

8. THE AXIOM SYSTEM MODEL FOR ASSERTION LOGIC

We now introduce a model for assertion logic. In this model an 'assertor' is a *consistent finite set of propositions* $x = \{p_1, p_2, \ldots, p_n\}$. Let $X = (p_1 \,\&\, \&p_2 \,\&\ldots\&\, p_n)$ be the conjunction of all the x-elements. Then we shall construe Axp to amount to $\vdash X \rightarrow p$, where '$\rightarrow$' represents a plausible mode of strict implication or entailment. Thus x asserts p if and only if p

is entailed by X. In effect, an assertor is now an arbitrary (but consistent) axiom system, and asserts whatever obtains in the system.

Note that the rule (R) is obviously satisfied: if $p \vdash q$, then $X \rightarrow p \vdash X \rightarrow q$. Axiom ($A1$) now asserts

$$(\forall X)\,(\exists p)\,[X \rightarrow p]$$

which is obviously true. Axiom (A2) is now

$$[(X \rightarrow p)\,\&\,(X \rightarrow q)] \rightarrow (X \rightarrow [p\,\&\,q])$$

which is also an obvious logical truth. Axiom (A3) becomes

$$\sim(X \rightarrow [p\,\&\sim p])$$

which follows at once from the consistency of X. Thus all of $\mathbf{A}_1$ will be satisfied by our model.

Moreover, the characteristic axioms of $\mathbf{A}_2$ and $\mathbf{A}_3$ are now

$$(\forall X)\,[X \rightarrow p] \rightarrow p$$

and

$$p \rightarrow (\exists X)\,[X \rightarrow p]$$

respectively. Both of these are also logical truths. Thus the axiom-system approach will provide a model for all of $\mathbf{A}_3$. Axiom (A_4) becomes

$$[X \rightarrow (X \rightarrow p)] \leftrightarrow (X \rightarrow p)$$

which will again be logically true.

However, the characteristic axiom of $\mathbf{A}_5$ will now be:

$$(\forall X)\,[(X \rightarrow p) \vee (X \rightarrow \sim p)].$$

Now this thesis would indeed obtain if our implication ($\rightarrow$) were to be as weak as material implication ($\supset$), but it will in general, fail for the stronger modes of implication such as C.I. Lewis' *strict implication* or *entailment* in the sense of Anderson-Belnap.

9. THE SYSTEM OF ŁOŚ

The founder of assertion logic is the Polish logician Jerzy Łoś. In an important 1948 paper[6], Łoś developed what he called a logic of 'belief'

[6] 'Logiki wielowartościowe a formalizacja funkcji intensyonalnych' [Many-Valued Logics and the Formalization of Intensional Functions], *Kwartalnik filozoficzny* **17** (1948) 59–87. I know the contents of this paper only from the brief report by A.N. Prior in his *Formal Logic* (Oxford, 1955) p. 313; and from two reviews, the first by H. Hiż in *Mathematical Review* **10** (1949) 1–2, and the other by R. Suszko in *The Journal of Symbolic Logic* **14** (1949) 64–65.

or 'assertion' upon the following basis. An Ł-operator was introduced, with '*Łxp*' to mean 'the man x believes (or: is committed to) the proposition p'. The following axioms are postulated to govern the formal logic of this idea:

(1) A group of axioms of the form
$Łx\alpha$, for all x (where α is an axiom of propositional logic)

(2) $Łx(\sim p) \equiv \sim Łxp$

(3) $Łx(p \supset q) \supset (Łxp \supset Łxq)$

(4) $Łx(Łxp) \supset Łxp$

(5) $(\forall x)\, Łxp \supset p$.

This system, **Ł**, is essentially equivalent with $\mathbf{A}_5$. Replacing Łoś's Ł by A it is not difficult to show the identity of the systems.

$\mathbf{A}_5$ may be derived from **Ł** as follows: (1) and (4) together yield (R). (1) yields (A1). (2) and (3) yield (A2). (1) and (2) yield (A3). (4) is identical with (A4). (5) yields (A5).

Ł may be derived from $\mathbf{A}_5$ as follows. (R*) yields (1). ($\mathbf{A}_4$:x) is identical with (2). (A1) and ($\mathbf{A}_5$:4) yields (3). (A5) is identical with (4). ($\mathbf{A}_2$) is identical with (5).

Thus the system **Ł** of Łoś *is equivalent with the logic of assertion for complete assertors*, i.e., with the system $\mathbf{A}_5$.[7]

10. THE RELATIONSHIP TO TOPOLOGICAL LOGIC AND FURTHER MODELS OF $\mathbf{A}_5$

In a recent paper[8] James Garson and the writer developed several systems of *topological logic*, based on the parametrized operator *Pxp* to be construed as 'the proposition p is realized at the position x'. One of these

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[7] According to the report in H. Hiż's review of his paper in *Mathematical Review*, *op. cit.*, Łoś regarded his system **Ł** (= our $\mathbf{A}_5$) to be interpretable as a logic of belief. Some rough analogy will certainly obtain between a logic of (rational) *assertion* (as based on *Axp*) and a logic of (rational) *belief* (as based on a similar operator *Bxp*). For example one will in both systems want the counterpart to:

$$\Phi x(p \,\&\, q) \equiv (\Phi xp \,\&\, \Phi xq).$$

But this identification of assertion logic with belief logic is hardly tenable in the face of the considerations adduced in Chap. V above.

[8] Nicholas Rescher and James Garson, 'Topological Logic', *The Journal of Symbolic Logic* **33** (1968). See Chap. XIII above.

systems, the system **PI**, rests on the following basis:

Axioms (P1) $Px(\sim p) \equiv \sim Pxp$
(P2) $Px(p \,\&\, q) \equiv (Pxp \,\&\, Pxq)$
(P3) $(\forall x)\, Pxp \equiv Py[(\forall x)\, Pxp]$
(P4) $(\forall x)\, Pxp \supset p$
(P5.1) $Py(Pxp) \equiv Pxp$, for all y (i.e., $(\forall x)\,[Pxp \equiv (\forall y)\, Py(Pxp)]$)
Rule If $\vdash p$ then $\vdash Pxp$.

It is readily shown (along lines already exploited in discussing the closely cognate system **Ł**) that this system, with (P3) deleted and (P5) simplified and restricted to

$$Px(Pxp) \equiv Pxp$$

is equivalent with $\mathbf{A}_5$. Assertion logic is thus a fragment of the system **PI** of topological logic. Indeed we can thus look upon the theory of complete assertors represented by the system $\mathbf{A}_5 = \mathbf{Ł}$ as a halfway house between the logic of assertion proper ($\mathbf{A}_1$), and the system **PI** of topological logic.[9]

This relationship can be exploited to yield some interesting models of assertion logic of which we shall consider only two.

Let the assertors x, y, z, etc. be elements of a set of 'possible worlds' $W = \{w_1, w_2, \ldots\}$. Let 'Pxp' obtain if and only if p is true in x. Since this model provides an interpretation for **PI**, it is then readily shown that all of $\mathbf{A}_5$ will be satisfied by this interpretation.

Again, let the assertors x, y, z, etc. be intervals of time. Let 'Pxp' obtain if and only if p is true for the interval x. Since this model provides an interpretation for **PI**, it is again readily shown that all of $\mathbf{A}_5$ will be satisfied by this interpretation.

The Rescher-Garson paper referred to above also describes (though it does not discuss in detail) a system of topological logic that is fundamentally 3-valued in that it does not postulate that 'Pxp' is invariably defined for every position x. This has the consequence that for some values of x, and for certain propositions p, 'Pxp' may be neither true nor false, but assume a third, non-classical truth-status (undefined, indeterminate, inapplicable, meaningless, neutral, or the like). The axioms for

[9] Since chronological logic (or 'tense logic') is a special type of topological logic – for details see Chap. XIII – this finding is also a step towards establishing the kinship of assertion logic with chronological logic.

this case will be:

(P0) $(Pxp \supset Pxq) \supset Px(p \supset q)$

(P1.1) $Px(\sim p) \supset \sim Pxp$

(P2), (P3), (P4), (P5.1) as above

(P6) $p \supset (\exists x)\, Pxp.$

That is, we add (P6) and trade in half of the old (P1) – viz. the converse of (P1.1) – against (P0), the same special rule of inference (If $\vdash p$ then $\vdash Pxp$) being retained as before. Apart from the stronger iteration principle – viz., (P5.1) in place of (A4) – this system is equivalent with the system $\mathbf{A}_3$ of assertion logic. Thus we can say that 3-valued topological logic differs from assertion logic primarily as regards its iteration principle. For throughout our development of assertion logic we have made no assumptions regarding $AyAxp$ with $y \neq x$.

11. A 3-VALUED PERSPECTIVE UPON ASSERTION LOGIC

Let us introduce the conception of 'the truth status of the assertion p with respect to the assertor x' – symbolically $/p/_x$ – in accordance with the following 3-valued approach:

$$/p/_x = \left\{\begin{matrix}\mathrm{T}\\ \mathrm{I}\\ \mathrm{F}\end{matrix}\right\} \text{ according as } \left\{\begin{matrix} Axp \\ \sim Axp \,\&\, \sim Ax \sim p. \\ Ax(\sim p)\end{matrix}\right.$$

Thus the truth value of a proposition p for a given assertor is true or false or indifferent according as he asserts that p or asserts that not-p or asserts neither that p nor that not-p. It should be noted that the indifferent case is at once ruled out for the case of a complete assertion logic where the axiom ($\mathbf{A}_5$) obtains.

Drawing upon the axioms of $\mathbf{A}_1$–$\mathbf{A}_3$ we shall derive the following 3-valued truth tables:

$/p/_x$	$/\neg p/_x$
T	F
I	I
F	T

$/p/_x$ \ $/q/_x$	$/p \wedge q/_x$			$/p \veebar q/_x$			$/p \rightarrow q/_x$		
	T	I	F	T	I	F	T	I	F
T	T	I	F	T	T	T	T	I	F
I	I	(I, F)	F	T	(I, T)	I	T	(I, T)	I
F	F	F	F	T	I	F	T	T	T

(We use $\neg$, $\wedge$, $\veebar$, and $\rightarrow$ as 3-valued counterparts to $\sim$, &, $\vee$, and $\supset$.) In some cases the truth-value is indeterminate because no *specific* derivation can be made. Thus when $/p/_x = I$ and $/q/_x = I$, then we must have:

$$\sim Axp \,\&\, \sim Ax(\sim p) \,\&\, \sim Axq \,\&\, \sim Ax(\sim q).$$

But what can we now say regarding $/p \,\&\, q/_x$? When (say) $p = q$, then the truth-value for x of the conjunct $p \,\&\, q$ will certainly be I. But when $p = \sim q$, then, since $Az \sim (q \,\&\, \sim q)$ is uniformly true, we shall have $/p \wedge q/_x = F$.

A set of truth-tables of this sort, some of whose entries are indeterminate, is said to be *quasi-truth-functional.*[10] It is thus an interesting feature of this approach that the truth-value structure of our assertion logic issue is *both many-valued and quasi-truth-functional.*

However, if the assertion system is *complete*, then the case of an indeterminate truth-value cannot arise, and then the assertion-logic collapses back into the classical two-valued propositional calculus.

12. ASSERTION LOGIC AND MANY-VALUED LOGIC

An interesting assertion-logical approach to many-valued logic goes as follows. Let the 'truth-value' of a proposition be *the set of all assertors who assert that proposition*:

$$/p/ = \{x | Axp\}.$$

The drawback of this approach is that we will in general obtain systems that are only quasi-truth-functional. For example, $/\neg p/$ may be any set between Λ and $/p/'$. We will thus in general not be able to give only *inequalities* for the truth-values of complex expressions, as follows:

$$\Lambda \subseteq /\neg p/ \subseteq /p/' \quad \text{(Here } ' \text{ represents set-complementation.)}$$
$$/p \wedge q/ = /p/ \cap /q/$$
$$/p/ \cup /q/ \subseteq /p \veebar q/ \subseteq V$$
$$/\neg p/ \cup /q/ \subseteq /\neg p \veebar q/ = /p \rightarrow q/ \subseteq V.$$

Thus suppose that there are just two assertors, x and y. There will then

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[10] See N. Rescher, 'Quasi Truth-Functional Systems of Propositional Logic', *The Journal of Symbolic Logic* 27 (1962) 1–10.

be four truth-values as follows:

$$1=\{x, y\}=V$$
$$2=\{x\}$$
$$3=\{y\}$$
$$4=\Lambda$$

The following quasi-truth-functional 4-valued system will result:

$\neg p$	p \ q	$p \wedge q$				$p \veebar q$				$p \rightarrow q$			
		1	2	3	4	1	2	3	4	1	2	3	4
4	1	1	2	3	4	1	1	1	1	1	(1,2)	(1,3)	(1,2,3,4)
(4,3)	2	2	2	4	4	1	(1,2)	1	(1,2)	1	(1,2)	(1,3)	(1,2,3,4)
(4,2)	3	3	4	3	4	1	1	(1,3)	(1,3)	1	(1,2)	(1,3)	(1,2,3,4)
(4,3,2,1)	4	4	4	4	4	1	(1,2)	(1,3)	(1,2,3,4)	1	(1,2)	(1,3)	(1,2,3,4)

Note that as long as 1 is a designated truth-value, this system – quasi-truth-functional though it is – will have various tautologies, specifically including '$\neg(p \wedge \neg p)$'.

Now the situation is substantially simplified if we assume that our basic assertion system is *complete* and thus has an $\mathbf{A}_5$-structure, so that:

$$Ax(\sim p) \equiv \sim Axp.$$

For then the 4-valued logic at issue is the (strictly truth-functional) system:

p	$\neg p$	p \ q	$p \wedge q$				$p \veebar q$				$p \rightarrow q$			
			1	2	3	4	1	2	3	4	1	2	3	4
1	4	1	1	2	3	4	1	1	1	1	1	2	3	4
2	3	2	2	2	4	4	1	2	1	2	1	1	3	3
3	2	3	3	4	3	4	1	1	3	3	1	2	1	2
4	1	4	4	4	4	4	1	2	3	4	1	1	1	1

This is identical with the system $\boldsymbol{C}_2 \times \boldsymbol{C}_2$, the Cartesian product of the classical two-valued propositional calculus with itself.[11]

In general, if an $\mathbf{A}_5$ structure is assumed, we will obtain a strictly truth-

[11] See A.A. Zinov'ev, *Philosophical Problems of Many-Valued Logic*, ed. and tr. by G. Küng and D.D. Comey (Dordrecht, 1963). Cf. p. 247 above.

functional system of many-valued logic subject to the truth rules:

$$/\neg p/ = /p/'$$
$$/p \wedge q/ = /p/ \cap /q/$$
$$/p \veebar q/ = /p/ \cup /q/$$
$$/p \rightarrow q/ = /\neg p \veebar q/ = /p/' \cup /q/.$$

When there are n assertors, the previous finding generalizes, and the many-valued logic that arises from $\mathbf{A}_5$ assertion logic will have the structure of:

$$\boldsymbol{C}_2 \times \boldsymbol{C}_2 \times \cdots \times \boldsymbol{C}_2 \ (n \text{ times}).$$

This approach to many-valued logic through assertion logic is due to Jerzy Łoś,[12] who must be credited with this particular finding that in the context of a complete assertion system (his system **Ł**) an n-assertor system will generate the n-factor self-product logic of $\boldsymbol{C}_2$.

13. MODALITY IN ASSERTION LOGIC

The system which Fitch characterizes as **DM** (the *deontic* version of G. H. von Wright's system **M**) is built up on the following basis:[13]

Primitive: $\boxed{\times}$ for *deontic necessity*, i.e., *obligation*
Rule: If $\vdash p$, then $\vdash \boxed{\times} p$
Axioms: (1) $\boxed{\times} \sim p \supset \sim \boxed{\times} p$
(2) $\boxed{\times} (p \supset q) \supset (\boxed{\times} p \supset \boxed{\times} q)$.

It should thus be noted that if the assertion operator Ax (for a fixed parameter-value x) is construed as a necessity-analogous modality, then the logical structure of the $\mathbf{A}_1$ of assertion logic is identical with that of **DM**: When 'Ax' and '$\boxed{\times}$' are identified, the axiomatic basis of the two systems is precisely the same.

Going beyond this finding, we may introduce modal operators into assertion logic by the definitions:

$$\Box p \quad \text{FOR} \quad (\forall x)\, Axp$$
$$\Diamond p \quad \text{FOR} \quad \sim \Box \sim p = \sim (\forall x)\, Ax(\sim p).$$

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[12] See Łos, *op. cit.*

[13] Frederic B. Fitch, 'Natural Deduction Rules for Obligation', *American Philosophical Quarterly* 3 (1966) 27–38.

Note that the thesis

$$\Box p \supset p$$

or equivalently

$$(\forall x)\, Axp \supset p$$

will not obtain in assertion logic in general, since it fails in systems weaker than $\mathbf{A}_2$. The principle at issue thus requires the characteristic axiom of $\mathbf{A}_2$ – and so fails for $\mathbf{A}_1$, while holding for $\mathbf{A}_2$–$\mathbf{A}_5$. Moreover,

$$p \supset \Diamond p$$

or equivalently

$$p \supset \sim(\forall x)\, Ax(\sim p)$$

is equivalent with

$$(\forall x)\, Axp \supset p$$

and so holds only when '$\Box p \supset p$' does. On the other hand

$$\Box p \supset \Diamond p$$

or equivalently

$$(\forall x)\, Axp \supset \sim(\forall x)\, Ax(\sim p)$$

and equivalently

$$(\forall x)\, Axp \supset (\exists x)\, A^*xp$$

will hold unqualified in all of the $\mathbf{A}_i$.

G.H. von Wright's initial system **M** has the following basis:

Rule: If $\vdash p$, then $\vdash \Box p$
Axioms: (1) $\Box p \supset p$
(2) $\Box(p \supset q) \supset (\Box p \supset \Box q)$
(3) $(\Box p \,\&\, \Box q) \supset \Box(p \,\&\, q)$.

It is thus clear that the modal structure of all the systems $\mathbf{A}_2$–$\mathbf{A}_5$ will satisfy the conditions of **M** (and *a fortiori* those of C.I. Lewis' system **S2**).

Another procedure for introducing modalities would be:

$$\diamondsuit p \quad \text{FOR} \quad (\exists x)\, Axp$$
$$\boxdot p \quad \text{FOR} \quad \sim\diamondsuit\sim p = \sim(\exists x)\, Ax(\sim p).$$

Note that in general:

$$\Box p \supset \boxdot p$$
$$\diamondsuit p \supset \Diamond p.$$

Thus $\Box$ and $\boxdot$ may be called strong and weak modes of necessity, respectively, while $\diamondsuit$ and $\Diamond$ may be called strong and weak modes of possibility, respectively.

To assure the coincidence of the two modes of modality, so that

$$\Box p \equiv \boxdot p \text{ and } \Diamond p \equiv \diamondsuit p$$

we need the added principle

$$(\forall x)\, A^*xp \supset (\forall x)\, Axp$$

or equivalently

$$(\exists x)\, A^*xp \supset (\exists x)\, Axp.$$

As we saw in Sect. 4 above this comes down to stipulating that the group of assertors at issue is *sceptical* – i.e., if every proposition not asserted by everyone is denied by someone:

(S) $$\sim(\forall x)\, Axp \supset (\exists x)\, Ax(\sim p).$$

Since the addition of this thesis to $\mathbf{A}_2$ will yield $\mathbf{A}_3$, we can say that the two modes of modality cannot coincide in an assertion system weaker than $\mathbf{A}_3$. It is perfectly obvious that they must coincide in $\mathbf{A}_5$.

Again, we shall not in general have the theses:

$$\boxdot p \supset p$$
$$p \supset \diamondsuit p.$$

Both of these are contingent upon having

$$\sim(\exists x)\, Ax(\sim p) \supset p$$

or equivalently (since we can throughout replace '$\sim p$' by 'p'):

$$p \supset (\exists x)\, Axp.$$

The principles at issue thus require the characteristic axiom of $\mathbf{A}_3$, and so will fail for $\mathbf{A}_1$–$\mathbf{A}_2$, while holding for $\mathbf{A}_3$–$\mathbf{A}_5$. However

$$\boxdot p \supset \diamondsuit p$$

will hold unqualifiedly in all of the $\mathbf{A}_i$.

To get beyond **M** (or **S2** in the series of Lewis systems) requires the addition of special axioms. Thus consider the characteristic axiom of **S4**:

$$\Box(\Box p \supset \Box\Box p).$$

To obtain this we need to establish (given the machinery of $\mathbf{A}_5$) that:

$$(\forall z)\, Az\{(\forall x)\, Axp \supset (\forall x)\, Ax[(\forall y)\, Ayp]\}.$$

We would thus have to prove:

$$(\forall x)\, Axp \supset (\forall x)\, Ax[(\forall y)\, Ayp].$$

In an $\mathbf{A}_5$-style system this is equivalent with

$$(\exists x)\, Ax[(\exists y)\, Ayp] \supset (\exists x)\, Axp$$

or equivalently

$$Az[(\exists x)\, Axp] \supset (\exists x)\, Axp.$$

That is, we would have to postulate that *only true assertions are made about the making of specific* assertions.

Again, consider the characteristic axiom of **S5**:

$$\Box(\diamondsuit p \supset \Box\diamondsuit p).$$

To obtain this we need to establish (given the machinery of $\mathbf{A}_5$) that:

$$(\forall z)\, Az[(\exists x)\, Axp \supset (\forall x)\, Ax\, [(\exists y)\, Ayp]].$$

We would thus have to prove

$$(\exists x)\, Axp \supset (\forall x)\, Ax[(\exists y)\, Ayp]$$

or equivalently

$$(\exists y)\, Ay[(\forall x)\, Axp] \supset (\forall x)\, Axp$$

or equivalently

$$Az[(\forall x)\, Axp] \supset (\forall x)\, Axp.$$

That is, we would have to postulate that *only true assertions are made about universal assertions.*

The situation regarding the modal systems induced by various species of assertion logic is surveyed in detail in Appendix III. The case may be summarized as follows:

1. The modal logic for a weak, $\mathbf{A}_2$-strength, assertion logic, as induced by its *undotted* assertion modalities, is von Wright's system **M** (and *a fortiori* C.I. Lewis' system S2).
2. To obtain an **M**-modal structure (or an S2 modal structure) from the *dotted* assertion modalities, we must go to a stronger, $\mathbf{A}_3$ strength, assertion logic.
3. To attain stronger modal systems (**B**, $\mathbf{S}_4$, $\mathbf{S}_5$) further special principles must be introduced into assertion logic, principles which result in systems even stronger than the system $\mathbf{A}_5$ for complete assertions. The additions which must be made relate essentially to the logical rules for the 'meta-assertions' which we shall consider in the next section.[14]

14. META-ASSERTIONS

By a 'meta-assertion' we understand an assertion about assertions. Some key examples are:

(1) $Ax(Axp)$
(2) $Ay(Axp)$ with $y \neq x$
(3) $Ay[(\exists x)\, Axp]$
(4) $Ay[(\forall x)\, Axp]$
(5) $(\exists y)\, Ay[(\exists x)\, Axp]$
(6) $(\forall y)\, Ay[(\exists x)\, Axp]$
(7) $(\exists y)\, Ay[(\forall x)\, Axp]$
(8) $(\forall y)\, Ay[(\forall x)\, Axp]$

The first of these, item (1), is, of course, postulated in $\mathbf{A}_4$–$\mathbf{A}_5$ to be reducible to Axp, and indeed the relevant axiom of redundancy, viz.,

$$Ax(Axp) \equiv Axp$$

is the somewhat meager foundation for the entire treatment of meta-assertions in our $\mathbf{A}_i$-style systems.

[14] For a summary of the details consult Appendix III.

We have made no assumption regarding the reducibility of (2). As was remarked in Sect. 11 above, just this is a key point of difference with topological logic, where the principle

$$Ay(Axp) \supset (\exists z)\, Azp$$

invariably obtains, and where, in the special case considered above, we have the axiom:

$$Ay(Axp) \equiv Axp.$$

It is not difficult to demonstrate that (8) is reducible to (i.e., equivalent with):

$$(\forall x)\, Axp.$$

Moreover, it is easy to show that (7) entails, but is not equivalent with:

$$(\exists x)\, Axp.$$

As we remarked in Sect. 13, one might (for special purposes) want to consider the special case of assertion systems obtained by stipulating that (7) is to imply:

$$(\forall x)\, Axp.$$

One would thus stipulate:

$$(\exists y)\, Ay[(\forall x)\, Axp] \supset (\forall x)\, Axp.$$

Again it is plausible that we should want to consider the special cases that (5) – and so also (3) – implies $(\exists x)\, Axp$:

$$(\exists y)\, Ay[(\exists x)\, Axp] \supset (\exists x)\, Axp.$$

This last-indicated stipulation would also have consequences of a modal variety, viz.,:

$$\boxdot(\diamondsuit\diamondsuit p \supset \diamondsuit p).$$

Thus addition of the stipulation in question to $\mathbf{A}_1$ underwrites an S4-like modal structure for the dotted modalities (along the lines already considered in the previous section).

It should be noted in this connection that we could introduce yet weaker modalities of possibility along alternative lines, such as

$$\diamondsuit p \quad \text{FOR} \quad (\exists y)\, Ay[(\exists x)\, Axp]$$

with the corresponding mode of necessity

$$\boxdot p \quad \text{FOR} \quad \sim\diamondsuit\sim p = (\forall y)\sim Ay[(\exists x)\, Ax(\sim p)]$$
$$\text{(equivalently: } (\forall y)\, A^*y[(\forall x)\, A^*xp]).$$

In $\mathbf{A}_5$ this mode of necessity will amount to

$$(\forall y)\, Ay[(\forall x)\, Axp]$$

which is, as we have seen, equivalent in general with

$$(\forall x)\, Axp.$$

Consequently in $\mathbf{A}_5$, the weaker and stronger modes of modality will again coincide, and will again yield a modal structure satisfying (*inter alia*) all the laws of **M**.

Certain meta-assertions will obviously have to be dismissed as self-contradictory, for example:

$$Ax[\sim(\exists p)\, Axp] \text{ or equivalently } Ax[(\forall p)\sim Axp]$$
$$Ax[(\forall p)\, Ax\sim p]$$
$$Ax[(\forall p)\, Axp].$$

The negations of each of these is demonstrable, so that each leads to a palpable contradiction. Thus consider:

$$(\exists x)\, Ax[\sim(\exists p)\, Axp].$$

Its negation is

$$\sim(\exists x)\, Ax[\sim(\exists p)\, Axp]$$

or equivalently

$$(\forall x)\sim Ax[\sim(\exists p)\, Axp]$$

or equivalently

(1) $$(\forall x)\, A^*x[(\exists p)\, Axp].$$

But

$$(\exists p)\, Axp$$

is a theorem in all the $\mathbf{A}_i$, so that this is also the case with

$$(\forall x)\, Ax[(\exists p)\, Axp]$$

from which (1) follows.

Again,

$$Ax[(\forall p)\, Axp]$$

can be refuted by *reductio ad absurdum*. For since $(\forall p)\, Axp \vdash Ax(q \,\&\sim q)$, it has the consequence

$$Ax[Ax(q \,\&\sim q)]$$

which entails (in $\mathbf{A}_4$ and $\mathbf{A}_5$, at any rate)

$$Ax(q \,\&\sim q).$$

And this contradicts the consistency axiom.

Thus the various self-contradictory meta-assertions can be refuted by means of the axioms.

15. ASSERTION AND PROPOSITIONAL FUNCTIONS

To this point we have considered only the assertion of complete propositions: our propositional variables 'p', 'q', 'r', etc. have ranged over self-contained propositions. A plausible step beyond this starting point is the introduction of propositional functions of and quantification over individuals proper. Thus let $D = \{a, b, \ldots\}$ be a set of individuals constituting the range of the variables 'ξ', 'η', etc., and let F, G, H, etc. be propositional functions of these individuals.

Quantificational logic alone assures us of the thesis:

(Q1) $\quad (\forall \xi)\, Ax(F\xi) \supset Ax(Fa).$

By means of the rule (R), one would, moreover obtain the thesis:

(Q2) $\quad (\exists \xi)\, Ax(F\xi) \supset Ax[(\exists \xi)\, F\xi].$

One would almost certainly want to adopt the following two theses:

(Q3) $\quad Ax(Fa) \supset (\exists \xi)\, Ax(F\xi)$

(Q4) $\quad Ax[(\forall \xi)\, F\xi] \supset (\forall \xi)\, Ax(F\xi).$

In $\mathbf{A}_5$ these will be equivalent with (Q1) and (Q2), respectively. Moreover, in terms of our proposed construction of the assertion operator, both of these theses would seem to be unqualifiedly acceptable.

Special situations are represented by two further restrictive (and not

unqualifiedly acceptable) theses, the converses of (Q4) and (Q5), respectively:

(Q5) $(\forall \xi)\, Ax(F\xi) \supset Ax[(\forall \xi)\, F\xi]$

(Q6) $Ax[(\exists \xi)\, F\xi] \supset (\exists \xi)\, [Ax(F\xi)]$.

The first of these theses makes the (contingent) claim that: *If x asserts of every ξ that it has F, then x asserts that every ξ has F.* This claim involves the (not implausible) requirement that every assertor be aware of the whole extent of the content of the domain *D*. The second thesis makes the (contingent) claim that: *If x asserts that some ξ has F, then x asserts of some particular ξ that it has F.* This is a claim which, taken in unqualified generality, would appear to be quite unacceptable. A plausible step is thus to consider the system arising from adoption of (Q1)–(Q5), but rejecting (Q6).

Given this machinery, it is easy to verify the acceptability of various standard theses of quantified modal logic. For example, consider the thesis:

$$\Box(\forall \xi)\, F\xi \supset (\forall \xi)\Box F\xi.$$

This now becomes

$$(\forall x)\, Ax[(\forall \xi)\, F\xi] \supset (\forall \xi)\, (\forall x)\, Ax(F\xi)$$

or equivalently

$$(\forall x)\, Ax[(\forall \xi)\, F\xi] \supset (\forall x)\, (\forall \xi)\, Ax(F\xi).$$

But this will obtain in view of (Q4). (Its converse would also obtain if we accepted (Q5).)

Special interest attaches to what has come to be known as the 'Barcan Formula':

$$\Diamond(\exists \xi)\, F\xi \supset (\exists \xi)\ \Diamond F\xi.$$

This now becomes

$$(\exists x)\, Ax[(\exists \xi)\, F\xi] \supset (\exists \xi)\, (\exists x)\, Ax(F\xi)$$

or equivalently

$$(\exists x)\, Ax[(\exists \xi)\, F\xi] \supset (\exists x)\, (\exists \xi)\, Ax(F\xi).$$

But this will be obtainable only if we stipulate the (unacceptable) thesis (Q6). (On the other hand, the converse of the Barcan Formula will obtain in view of (Q2).)

16. INCONSISTENT ASSERTORS: THE SYSTEM $\mathbf{A}_0$

It is of interest to inquire into the situation that results when the requirement of consistency is dropped, and the corresponding axiom (A3) abandoned. We now obtain a subsystem of $\mathbf{A}_1$, let us call it $\mathbf{A}_0$, built up on the following basis:

(1) $(\forall x)(\exists p)\, Axp$

(2) $Ax(p \supset q) \supset (Axp \supset Axq)$

(R) If $p \vdash q$ then $Axp \vdash Axq$.

We shall again have the derived rule:

(R*) If $\vdash p$ then $\vdash Axp$.

And we shall have the theorems:

$$Ax(p \,\&\, q) \equiv (Axp \,\&\, Axq)$$

$$(Axp \vee Axq) \supset Ax(p \vee q).$$

(Alternatively, we could base an equivalent system upon (R*) and axiom (2) alone.)

This system of assertion logic has a very interesting application, because it enables us to circumvent the semantical paradoxes in which indirect discourse is involved.[15] It is important to discriminate between: (i) The occurrence of a paradox or a self-contradiction in our own logical theory of assertion statements (which would be fatal), and (ii) the assertion by someone (else) of a paradoxical or self-contradictory statement (which is quite possible, and harmless to our theory). Consider, for example, the statement: $Ax(p \,\&\sim p)$. By (R), this entails Axp and $Ax(\sim p)$, and indeed is Axq, for every and any q we please. Thus x has, to be sure *contradicted himself*, and has become a degenerate – because utterly indiscriminating – assertor. But

(i) $Axp \,\&\, Ax(\sim p)$

unlike

(ii) $Axp \,\&\sim Axp$

[15] Much of the ensuing discussion has been drawn from N. Rescher, 'Semantic Paradoxes and the Propositional Analysis of Indirect Discourse', *Philosophy of Science* **28** (1961) 437–440.

does not now (i.e., in $\mathbf{A}_0$) engender any self-contradiction in *our* assertions.

The paradoxical situations of Type (i) – unlike those of Type (ii) – are completely harmless for the propositional analysis of indirect discourse, because they intrude *another assertor* between the paradoxical statements and the statements of our discussion.[16]

To see how this assertion-logical approach, with its use of a propositional analysis of indirect discourse, can circumvent semantical paradoxes that arise in this sphere let us consider three typical semantical paradoxes in which indirect quotation plays a central role:

(I) Epimenides the Cretan asserts that all Cretans are (always) liars.

(II) Mr. x, the notorious selfcontradictor, asserts that whatever he himself asserts is false.

(III) x asserts that this selfsame assertion of his is false.

Derivation of the 'Paradoxical' Consequence of (I)

1.	$Ae[(\forall x)(\forall p)[(Cx \& Axp) \supset \sim p]]$	Premiss
2.	Ce and indeed even $Ae(Ce)$	Premiss
3.	Aep_1 where $p_1 = (\forall x)(\forall p)[(Cx \& Axp) \supset \sim p]$	From 1
4.	$Ae[Ce \& Aep_1] \supset Ae(\sim p_1)$	From 1[17]
5.	$Ae(Aep_1)$	From 3[18]
6.	$Ae(Ce \& Aep_1)$	From 2, 5
7.	$Ae(\sim p_1)$	From 4, 6
8.	$Ae(p_1 \& \sim p_1)$	From 3, 7

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[16] The saving feature is the *noncommittal* character of the assertion-relationship A. If A were such that the maintenance of Axp committed *us* to asserting p – if, for example, we had: $Axp \supset p$ – then paradox would be upon us.

[17] Note that this step represents an inference of the form:

$$\frac{(\forall \mathrm{p})\, \Phi \mathrm{p}}{\therefore \Phi(\forall \mathrm{p})\, \Phi \mathrm{p}}.$$

This mode of inference presupposes the absence of a self-reference excluding hierarchy principle, such as the theory of types, which would prevent us from including within the range of a propositional variable of a statement that very statement itself. We shall return below to the significance of this procedure.

[18] We assume here the availability of (A_4), wishing to give maximal scope to the paradox-deriver.

Derivation of the 'Paradoxical' Consequence of (II)

1.	$Ax[(\forall p)(Axp \supset \sim p)]$	Premiss
2.	Axp_1 where $p_1 = (\forall p)(Axp \supset \sim p)$	From 1
3.	$Ax(Axp_1 \supset \sim p_1)$	From 1
4.	$AxAxp_1 \supset Ax(\sim p_1)$	From 3
5.	$Ax(Axp_1)$	From 2[19]
6.	$Ax(\sim p_1)$	From 4, 5
7.	$Ax(p_1 \& \sim p_1)$.	From 2, 6

Derivation of the 'Paradoxical' Consequence of (III)

1.	$Ax[p_1 \&(\forall q)([q = p_1] \supset \sim p_1)]$	Premiss
2.	$Ax(p_1 \& \sim p_1)$	From 1

Thus in each case, the paradox at issue is not of the logically intolerable form

$$p \& \sim p$$

but of the type

$$Ax(p \& \sim p).$$

And such 'paradoxical' results are harmless for our own propositional analysis of indirect discourse, because *another assertor* has been interjected as buffer between our own statements made *in propria persona*, and the selfcontradictory statements at issue.

A similar but more complex (and so more interesting) situation arises when the paradox derives from the declarations of two assertors. For example:

> Mr. x asserts that whatever Mr. x asserts is false, while y asserts that whatever x asserts is true.

Here we can derive a 'paradox' as follows:

1.	$Ax[(\forall p)(Ayp \supset \sim p)]$, and indeed even suppose $Ay[Ax[(\forall p)(Ayp \supset \sim p)]]$	Premiss
2.	$Ay[(\forall p)(Axp \supset p)]$, and indeed even suppose $Ax[Ay[(\forall p)(Axp \supset p)]]$	Premiss
3.	Ayp_1 and even $Ax(Ayp_1)$, where $p_1 = (\forall p)(Axp \supset p)$	From 2

[19] We again assume the availability of (A_4).

4.	$Ax[Ayp_1 \supset \sim p_1]$	From 1
5.	$Ax[Ayp_1 \supset \sim(\forall p)\,(Axp \supset p)]$	From 4
6.	$Ax[Ayp_1 \supset (\exists p)\,(Axp \,\&\sim p)]$	From 5
7.	$Ax(Ayp_1) \supset Ax[(\exists p)\,(Axp \,\&\sim p)]$	From 6
8.	$Ax[(\exists p)\,(Axp \,\&\sim p)]$.	From 3, 7

Here (8) is as close as we can come to the more paradoxical

$$(\exists p)\,Ax[Axp \,\&\sim p] \equiv (\exists p)\,[Axp \,\& Ax(\sim p)] \equiv$$
$$\equiv (\exists p)\,Ax(p \,\&\sim p)$$

which, however, is – as we saw above – still essentially harmless.

It appears, therefore, that *on a propositional analysis of indirect discourse in a suitable system of assertion logic, no special conventions to rule out self-reference are required to avoid the semantic paradoxes in which indirect quotation is involved.* The very manner of the analysis is itself sufficient to assure that the semantical paradoxes involving indirect discourse do not lead to damaging results, even in the absence of any hierarchy principle to avoid 'vicious' self-reference. Of course, if the assertor at issue is a logical system of some kind, a hierarchy principle may be needed to keep *it* from self-contradiction. But this fact does not militate against the dispensability of such a principle for *our* analysis of indirect discourse.[20]

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[20] I am grateful to Miss Sandy Roper for help in checking some of the arguments of this chapter, in the preparation of which I have benefited from comments upon a draft version by Nuel D. Belnap, Jr., Joseph Camp, James Garson, and Storrs McCall.

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APPENDIX I

THE SYSTEMS $\mathbf{A}_1$–$\mathbf{A}_5$ OF ASSERTION LOGIC

A standard system of propositional and quantificational logic is presupposed, with the rules of substitution and *modus ponens* assumed as rules of inference. To this basis we add the assertion operator Axp subject to the special rule:

(R) If $p \vdash q$ then $Axp \vdash Axq$. [Commitment]

Axioms for the basic system $\mathbf{A}_1$:

(A1) $(\forall x)(\exists p)\, Axp$ [Nonvacuousness]
(A2) $(Axp \,\&\, Axq) \supset Ax(p \,\&\, q)$ [Conjunction]
(A3) $\sim Ax(p \,\&\sim p)$ [Consistency]

$\mathbf{A}_2$:$\mathbf{A}_1$ plus

($\mathbf{A}_2$) $(\forall x)\, Axp \supset p$ [Lincoln]

$\mathbf{A}_3$:$\mathbf{A}_1$ plus

($\mathbf{A}_3$) $p \supset (\exists x)\,(Axp)$ [Collective Omniscience]

$\mathbf{A}_4$:$\mathbf{A}_1$ plus ($\mathbf{A}_3$) plus

($\mathbf{A}_4$) $AxAxp \equiv Axp$ [Redundancy]

$\mathbf{A}_5$:$\mathbf{A}_1$ plus ($\mathbf{A}_4$) plus

($\mathbf{A}_5$) $Axp \vee Ax(\sim p)$. [Completeness]

The following provides an alternative basis for $\mathbf{A}_1$:

(R*) If $\vdash p$ then $\vdash Axp$.

Axioms: (1) $Ax(\sim p) \supset \sim Axp$
(2) $Ax(p \supset q) \supset (Axp \supset Axq)$.

For the system $\mathbf{A}_0$, omit axiom (1) here.
The following provides an alternative basis for $\mathbf{A}_5$:

(R*) If $\vdash p$ then $\vdash Axp$.

Axioms: (1) $Ax(\sim p) \equiv \sim Axp$
(2) $Ax(p \,\&\, q) \equiv (Axp \,\&\, Axq)$
(3) $Ax(Axp) \equiv Axp$.

THE MODAL LOGICS INDUCED BY ASSERTION LOGIC

Basic Definitions: A^*xp FOR $\sim Ax(\sim p)$
$\Box p$ FOR $(\forall x)\, Axp$
$\Diamond p$ FOR $\sim\Box\sim p = (\exists x)\, A^*xp$
$\dot{\Diamond} p$ FOR $(\exists x)\, Axp$
$\dot{\Box} p$ FOR $\sim\dot{\Diamond}\sim p = (\forall x)\, A^*xp$.

Case I: The Undotted Modalities

1. We have **DM** once we have $\mathbf{A}_1$.
2. To get **M** we must add to $\mathbf{A}_1$:

$$(\forall x)\, Axp \supset p.$$

That is, we must go to $\mathbf{A}_2$.

3. To get **B** we must add to $\mathbf{A}_1$

$$p \supset (\forall y)\, Ay\,[(\exists x)\, Axp]$$

or equivalently

$$(\exists y)\, Ay\,[(\forall x)\, Axp] \supset p.$$

4. To get **S4** we must add to $\mathbf{A}_1$:

$$(\forall x)\, Axp \supset (\forall y)\, Ay\,[(\forall x)\, Axp].$$

5. To get **S5** we must add to $\mathbf{A}_1$:

$$(\exists y)\, Ay\,[(\forall x)\, Axp] \supset (\forall x)\, Axp.$$

Case II: The Dotted Modalities

1. We have **DM** once we have $\mathbf{A}_1$.
2. To get **M** we must add to $\mathbf{A}_1$:

$$p \supset (\exists x)\, Axp.$$

That is, we must go to $\mathbf{A}_3$.

3. To get **B** we must add to $\mathbf{A}_1$

$$(\exists y)\, Ay\,[(\forall x)\, A^*xp] \supset p$$

or equivalently

$$p \supset (\forall y)\, A^*y\,[(\exists x)\, Axp].$$

4. To get **S4** we must add to $\mathbf{A}_1$:

$$(\exists y)\, Ay\,[(\exists x)\, Axp] \supset (\exists x)\, Axp.$$

5. To get **S5** we must add to $\mathbf{A}_1$:

$$(\exists x)\, Axp \supset (\forall y)\, Ay\,[(\exists x)\, Axp].$$

A SUMMARY OF MODAL SYSTEMS

Throughout, substitution, *modus ponens*, and replacement of provable equivalents are used as rules of inference. Necessity $\Box$ is primitive throughout.

1. The System **M** (G.H. von Wright, 1951 as simplified by R. Feys).

   Rule: If $\vdash p$, then $\vdash \Box p$
   Axioms: (1) $\Box p \supset p$
   (2) $\Box(p \supset q) \supset (\Box p \supset \Box q)$

   Note 1: **M** is a strengthening of Lewis' **S2**, where the applicability of the rule *If* $\vdash p$, *then* $\vdash \Box p$ is restricted to *unmodalized* propositions p.

   Note 2: For the system **DM** (F. B. Fitch, 1966), weaken Axiom (1) to:

   $$\Box p \supset \sim \Box \sim p$$

   Note 3: For C.I. Lewis' system **S3**, strengthen Axiom (2) to:

   $$\Box(p \supset q) \supset \Box(\Box p \supset \Box q)$$

2. The System **B** (The *Brouwerian* System)
   To **M** add: $\Diamond \Box p \supset p$ (or equivalently: $p \supset \Box \Diamond p$).

3. The System **S4** (C.I. Lewis, 1932)
   To **M** add: $\Box p \supset \Box \Box p$.

   Note: **S4** neither contains nor is contained in **B**

4. The System **S5** (C.I. Lewis, 1932)
   To **M** add: $\Diamond \Box p \rightarrow \Box p$.
   Note: **S5** contains both **B** and **S4**.
   Summary of relationships:

$$\mathbf{S2} \supset \mathbf{M} \supset \left\{\begin{matrix}\mathbf{B} \\ \mathbf{S4}\end{matrix}\right\} \supset \mathbf{S5}.$$

For a fuller discussion of these systems see A. N. Prior, *Formal Logic* (Oxford, 1955), especially pp. 305–307, and Frederic B. Fitch, 'Natural Deduction Rules for Obligation', *American Philosophical Quarterly* **3** (1966) 27–38.

CHAPTER XV

THE LOGIC OF PREFERENCE

1. HISTORICAL INTRODUCTION

The founder of the 'logic of preference' is the founding father of logic itself, Aristotle. Book III of the *Topics* – where Aristotle is concerned to spell out principles governing the concept of preferability (αἱρετῶτερον = the worthier of choice) – must be regarded as the inaugural treatment of the subject. The treatment there, however, is such that no adequate distinction is drawn between material and formal considerations. The bulk of the principles listed are of a strictly substantive, non-formal sort. For example:

> That which is more permanent or durable is preferable to that which is less so. (116a13–14)
>
> That which is to be chosen for its own sake is preferable to that which is to be chosen for the sake of another. (116a29–30)

Other principles given are of a more formal and logically more tractable sort. For example:

> The possible (practicable) is preferable to the impossible (impracticable). (316b27)
>
> That is preferable which is the more applicable on every occasion or on most occasions (for example, justice and self-control are preferable to courage, for the first two are always applicable, but courage only sometimes). (117a35–37)
>
> If *A* be absolutely better than (preferable to) *B*, then also the best specimen of *A* is better than (preferable to) *B*; e.g., if man is better than horse, then also the best man is better than the best horse. (117b33–35)

The study of preference-principles acceptable upon abstract, formal, systematic grounds rather than upon any particular substantive theory of

preferability-determination is the task which the philosophically oriented 'logic of preference', as we envisage it, is to set for itself.

In recent philosophy, this logical enterprise was revived in the orbit of influence of the school of Brentano – particularly by Hermann Schwarz and Max Scheler.[1] The relevant work of this school has been carried forward by several continental investigators before World War II,[2] and since that time this line of inquiry has flourished especially in Scandinavia.[3] Only recently has the subject begun to arouse interest in the U.S.A. – a phenomenon for which R. M. Martin and R. M. Chisholm have been largely responsible.[4]

Interest on the part of economists in the theory of preference as a special application of the concept of utility long antedates this philosophical tradition. For the traditional theory on the economists' side of the border, the reader should consult Ch. VI, 'Value and Utility', of Alfred Marshall's classic *Principles of Economics*.[5] The recent formal development of this utility concept has primarily been in the direction of the mathematical theory of games,[6] in the wake of the pioneering work of von Neumann and Morgenstern.[7] The concept of valuation (and thus of preference) also plays a prominent and illuminating role in modern decision-theory.[8]

It is a central part of the motivation of this paper to try to develop the logic of preference in such a way as to build a bridge connecting these traditions: the logico-philosophical on the one hand, and the mathematico-economic on the other.

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1 See SCHWARZ (1900) and SCHELER (1913–16). For all citations of this sort see the Bibliography at the end of this chapter.

2 See KATKOV (1937) and KRAUS (1937).

3 See HALLDÉN (1957), VON WRIGHT (1963), and AQVIST (1963).

4 See MARTIN (1963), CHISHOLM (1964), and CHISHOLM and SOSA (1966).

5 London, Macmillan and Co., 1890; 8th ed., 1920. For a helpful history of the recent history of economic utility theory see D. Braybrooke, 'Farewell to the New Welfare Economics', *Review of Economic Studies* **23** (1955) 180–193. A bibliography of economic utility theory as well as philosophical utilitarianism is given in N. Rescher, *Distributive Justice* (New York, 1966).

6 See R. D. Luce and H. Raiffa, *Games and Decisions* (New York, Wiley, 1957), where a comprehensive bibliography is also given.

7 John von Neumann and Oskar Morgenstern, *Theory of Games and Economic Behavior* (Princeton, 1944; 2nd. ed., 1947).

8 See R. Jeffreys, *The Logic of Decision* (New York, McGraw-Hill, 1965).

2. MODES OF PREFERENCE

1. *Two Modes of Goodness*

The locutions 'it is a good thing that p' or 'p's being the case is a good thing' – involve a serious ambiguity in the notion of 'good' at issue. Certain essential distinctions must be drawn if confusion is to be avoided. The need for these distinctions becomes manifest if we examine some specific illustrative situations. Consider, for example, the following case:

If it is the case that	then one is to get
p	+ \$1
not-p	(unspecified).

Here p's being the case is a 'good thing' in the precise sense that, supposing p to be so, a positive result ensues. (Locutions of the type 'there is something in it for me' and 'I am better off than I was before' come to be applicable.) The basis of comparison here is that between:

(1) my situation *before* it eventuated that p was the case, and

(2) my situation *after* the eventuation of p.

Correspondingly, by this standard of comparison, the p of the following situation

If it is the case that	then one is to get
p	− \$1
not-p	(unspecified)

is a bad thing, again in the precise sense that when p comes about 'I stand to lose something' and 'I am worse off than I was before'.

Let us call this mode of goodness and badness – as based on a straightforward comparison of (1) and (2) – *first-order* goodness and badness, choosing this qualification because the mode of 'goodness' at issue turns on the value of the result of only one possible alternative in the situation in question (viz., p's being the case) to the exclusion of any concern with the result of the other alternative (viz., p's not being the case).[9]

That considerations over and above those involved in first-order good-

[9] This conception would, I suppose, come reasonably close to capturing some aspects of the intent of the traditional notion of *intrinsic* goodness, but this idea has been used in such a variegated (and sometimes loose) way that it is pointless to employ it for our purposes.

ness (or badness) must be introduced is readily established by the example of situations such as the following:

If it is the case that	then one is to get
p	+ \$1
not-p	+ \$10.

By the standards of comparison (1) and (2) laid down above, it is clear that *both* p's and not-p's coming to be the case are first order 'good things'. Nevertheless, it is also plain that one is entitled to regard p's happening as in some (different) sense a 'bad thing' because p's happening of course precludes not-p's happening, and not-p's happening is – under the circumstances – a good thing of such a magnitude that one cannot but regard its preclusion as a 'bad thing'.

The contrast at issue here is brought out even more explicitly in the following situation:

If it is the case that	then one is to get
p	− \$1
not-p	− \$100.

Neither p nor not-p are first order 'good things'. Yet one cannot, under the circumstances, regard p's happening as other than a very 'good thing' indeed, since it averts the minor catastrophe that ensues when not-p.

The contrast at issue in such cases, then, is not that between (1) and (2) as above, but that between

(2) my situation after the (assumed) eventuation of p, and

(3) my situation after the (assumed) eventuation of not-p.

We shall call the type of goodness at issue here *differential* goodness. It is assessed by taking into *comparative* account – not just the situation under consideration, but – the possible *alternatives* to the situation under consideration. As our examples have shown, it is clear that there is no inevitable correlation between first-order goodness (and badness) and differential goodness (and badness). First-order goods can be differential evils, and conversely.

2. *The Two Corresponding Modes of Preference*

As one would suspect, the distinction between *first-order* and *differential* goodness carries over into a parallel distinction between two correspond-

ing modes of preference. This is again brought out most sharply by an examination of examples. Consider the following situation where its coming-about-that-p and its coming-about-that-q are assumed to be independent events:

If it is the case that	then one is to get
p	+ \$10
not-p	(unspecified)
q	+ \$1
not-q	(unspecified).

Here one obviously prefers p's being the case to q's being the case, in the precise sense that the former conduces more to my benefit (i.e., is such that one 'stands to gain more by it'). This point can be made in the following terms: that the *extent* of p's first-order goodness (viz., a 'gain' of \$10) is greater than the extent of q's first-order goodness (viz., a 'gain' of \$1). We shall designate the mode of preferability at issue here as *first-order preference.* This mode of preference then is based upon a contrast of *the comparative extent of the first-order goodness* of the two items being compared.

Consider now the contrasting situation:

If it is the case that	then one is to get
p	+ \$2
not-p	+ \$2
q	+ \$1
not-q	− \$100.

The significant features to be noted with respect to this situation are: (i) There is no question that p is first-order preferable to q (the first order goodness of p stands to that of q in a ratio of 2:1). Nevertheless (ii) it is a matter of genuine indifference to oneself whether p is or is not the case – exactly the same result accrues either way. Moreover, (iii) it is highly important to one that q rather than not-q be the case (more than \$100 being at stake with respect to these alternatives). In view of this – and despite the first-order preferability of p to q – there is a definite and important sense in which we prefer q's being the case to that of p. This point can perhaps be sharpened by retabulating the alternatives as follows:

If it is the case that	then one is to get
$p \& q$	+ \$3
$p \& \sim q$	− \$98
$\sim p \& q$	+ \$3
$\sim p \& \sim q$	− \$98.

Our intuitive preference for q's being the case in contrast to p's being the case is based on the impressive contrast between the facts that, as this tabulation renders palpable:

1. When q is the case, one cannot fail to gain \$3 regardless of whether p is or is not the case.
2. When p is the case, then one either gains \$3 or *loses* \$98 depending upon whether q is or is not the case.

The sort of preference at issue here is clearly based upon a contrast of *the comparative extent of the differential goodness* of the two items being compared. Correspondingly, this type of preference will be designated as *differential preference.*

Our aim in this chapter will be to lay the groundwork for a detailed study of the logical theory of preference relationships of this general sort: to elucidate what is involved in preference-commitments of various types. The question of special interest to the economists – viz., how to *combine* the preference-commitments of diverse individuals into one coherent interpersonal preferential scheme – lies outside the purview of our discussion.

3. SEMANTICAL MACHINERY

1. *The Line of Approach*

The basis for the semantical considerations we are attempting to develop is the concept of a propositional preference ordering. Presupposing a propositional logic of the familiar sort – and representing propositions by the meta-variables 'α', 'β', 'γ', etc. – we introduce the propositional relationship P (for preferability) with the understanding that

$$\alpha P \beta$$

is to be understood as 'α's being the case is preferred (preferable) to β's being the case'. The only indispensable requirements we shall impose on the relationship P are:

1. That P be an ordering relation, i.e., that it be transitive, asymmetric, and irreflexive.
2. That P be an extensional relation among propositions, i.e., that it admit the substitution of provable equivalents.

The exact interpretation of the sort of preference at issue in P is to be left open. Specifically we shall not try to settle whether this is to be

(1) A matter of being *preferred* by a given individual (or group),

or

(2) A matter of being *preferable* by some impersonal criterion.

Nor shall we make any specification as to whether the preference at issue is a *synoptic* preference ('preferable when everything is taken into account') or an *aspectival* preference ('preferable in point of cost, or convenience, or the like'). However important these distinctions may be of themselves, they should be indifferent to the sort of abstract and 'structural' considerations to which a *logic of preference* devotes itself.

* * *

There are two alternative approaches to the development of a logic of preference: the *axiomatic* and the *semantic*. On the axiomatic approach, one lays down certain *basic formalized rules* – presumably underwritten by intuitive considerations – as guiding basis for the formal development of a theory. From these basic rules the theory itself is then derived as a logical consequence. On the semantic approach one sets up a *criterion of acceptability* for such rules and includes in one's system all those rules classed as acceptable by the criterion. The former, axiomatic approach has to date been the standard for the logic of preference. The systematizations of von Wright, Halldén, Chisholm-Sosa, and Martin have all proceeded in its purview. The approach is, however, unsatisfactory because of the wide divergence among these pioneers as to just what the 'obviously acceptable' principles of a logic of preference are. (Only the irreflexivity, asymmetry, and transitivity of preference lie in the range of the clearly unproblematic.) We ourselves shall pursue various *alternative but in principle reasonable* possibilities along the line of the semantical approach. Here divergences are less harmful and issue not in outright inconsistencies of a system, but in alternative plausible specifications of one intrinsically ambiguous idea. The approach is an experimental one: we do not seek to find 'the correct' logic of preference but to explore some of the more

promising systems that can be built up by way of tracing out various, of themselves plausible, conceptions of the nature of the preference relation. We hope in this way to be able to provide a *rationale* capable of explaining the divergencies between the several mutually incompatible axiomatizations that have been proposed for preference-logic by various recent writers on the subject.

2. *Formal Machinery of Analysis: Semantical Considerations*

To develop the semantical groundwork of a theory that deserves the name of a 'logic of preference', the sorts of considerations with which we have been dealing must be provided with a somewhat more systematic grounding.

Assume that we have a list

$$w_1, w_2, \ldots, w_n$$

of 'possible worlds' (state descriptions in the sense of Carnap). Our starting-point is provided by an *index of merit* measure $\sharp$ which assigns to each possible world w_1 a real-number value $\sharp(w_1)$. We shall not now enter upon a discussion of the specific substantive character of this measure, i.e., the specific respects in which it assesses the characteristics of the possible worlds, the sorts of considerations of which it takes account, and the relative importance with which it endows them. Considerations of this sort relate to the material side of the concrete application of the machinery and not to the formal side of the abstract logic of the matter, which alone concerns us at present.

Given this measure-of-merit or desirability for the possible worlds, we then assign to any proposition α that can be generated as a truth-functional compound of the w_1 the real number value $\sharp(\alpha)$ to be *the average (arithmetical mean) of the $\sharp$-values of all the possible worlds w_1 within which α is true.*[10] (This leaves $\sharp(\alpha)$ undefined when α is a contradiction; a difficulty which we shall simply lay aside for the time being.)[11]

[10] An interesting variant of this approach would be to consider a distribution of probabilities across the possible worlds w_i and then consider the correspondingly weighted average of the w_i, rather than their arithmetical mean. This approach is adopted as standard in Jeffrey's monograph [see JEFFREY (1965), Ch. 5], where, however, the propositional logic of the situation is not worked out. We shall return to the matter in footnote 17 below.

[11] The reader, can, if he likes, remove this gap by thinking of $\sharp(a)$ as fixed at 0 in this case.

We may construe $\#(\alpha)$ as measuring the extent of the first-order goodness of α (i.e., of the circumstance of α's being the case). And in this case the corresponding mode of preference will obviously be represented by the definition:[12]

$$\alpha P^{\#} \beta \text{ for } \#(\alpha) > \#(\beta).$$

Our $\#$-measure for first-order goodness is readily applied to a derivative measure for differential goodness – let it be represented by $\star$:

$$\star(\alpha) = \#(\alpha) - \#(\sim \alpha).$$

This measure has the interesting feature, which is of far-reaching significance – and whose counterpart emphatically does not hold for $\#$ – that:

$$\star(\sim \alpha) = - \star(\alpha).$$

The corresponding mode of preference will obviously be represented by the definition:

$$\alpha P^{\star} \beta \text{ for } \star(\alpha) > \star(\beta).$$

For the sake of an illustration of this group of ideas, consider the following:

Possible worlds	$\#$-values
$w_1: p \,\&\, q$	a
$w_2: p \,\&\sim q$	b
$w_3: \sim p \,\& q$	c
$w_4: \sim p \,\&\sim q$	d.

On this basis we may calculate, by way of illustration:

$$\#(p) = \frac{a+b}{2} \qquad \star(p) = \frac{a+b}{2} - \frac{c+d}{2}$$

$$\#(\sim p) = \frac{c+d}{2} \qquad \star(\sim p) = \frac{c+d}{2} - \frac{a+b}{2}$$

---

[12] Note that we must throughout *exclude substitutions* that make α or β into *contradictions*, once we have left $\#$ undefined in this case.

$$\#(q) = \frac{a+c}{2} \qquad \star(q) = \frac{a+c}{2} - \frac{b+d}{2}$$

$$\#(\sim q) = \frac{b+d}{2} \qquad \star(\sim q) = \frac{b+d}{2} - \frac{a+c}{2}$$

$$\#(p \vee q) = \frac{a+b+c}{3} \qquad \star(p \vee q) = \frac{a+b+c}{3} - d$$

$$\#(p \,\&\, q) = a \qquad \star(p \,\&\, q) = a - \frac{b+c+d}{3}$$

Thus, to say that $pP^{\#}q$ would be to have it that:

$$\frac{a+b}{2} > \frac{a+c}{2} \text{ or } b > c.$$

Or again, to say that $(p \vee q)\, P^{\star} p$ would be to have it that:

$$\frac{a+b+c}{3} - d > \frac{a+b}{2} - \frac{c+d}{2} \text{ or } 5c > a + b + 3d.$$

However, the salient difference between these two modes of preference is brought out by the fact that the principle

$$pPq \rightarrow \sim qP \sim p$$

which fails in general for $P^{\#}$ does hold for $P^{\star}$. (See the discussion of the principle (R1) in Sect. IV below.)

(A mode of preference with which it would also be interesting to deal is *preferability-other-things-being-equal*, that is, a relation P' such that (say):

> $pP'q$ iff $\#(p \,\&\, r) > \#(q \,\&\, r)$ whenever r is independent of p and of q.

Although this sort of conception can be handled with the machinery here introduced, its treatment involves additional complications which militate against our dealing with it here. Compare, however, the treatment of the principle (W5) in Sect. 4 below.)

Viewed in somewhat general terms, our approach to the logic of preference thus proceeds in terms of a numerical criterion of merit. Given a proposition α, we determine in some suitable way a numerical *measure of merit* $\mu(\alpha)$. And then we introduce a corresponding preference relation

P^{μ} with the convention that:

$$\alpha P^{\mu} \beta \text{ for } \mu(\alpha) > \mu(\beta).$$

3. *A Purely Qualitative Alternative Approach*

To the mind of some readers, the preceding quantitative line of approach might seem to have an air of unrealistic oversophistication. To assign to each possible world a specific real-number as its 'measure of merit' might appear a procedure that presupposes an unattainable differentiation in degrees of value. This line of criticism could, however, be accepted without vitiating the strategy of approach. We reply to the critic: 'Have it your way – don't even try for precise distinctions! Grade possible worlds into (say) just three classes: desirable, undesirable, and neutral. You can still apply – and benefit from – the application of the machinery here constructed'.

The fact is clear that our entire procedure could be carried on in the setting of just three rough, individually undifferentiated entries of an index of merit measure. Merely let the w_1 assume just one of three ♯-values, as follows:

+ 1 favorable (desirable)
0 neutral
– 1 unfavorable (undesirable)

The whole of the semantical machinery we shall construct now can be applied on the basis of such very rough and unsophisticated purely qualitative merit-assessments – assessments so crude and rough-cut[13] that even the reader favorably inclined to the sentiments of our hypothetical critic, could hardly demur from following the direction of our quantitative method for so slight a distance.

* * *

It must, however, be made clear that our semantical approach to the logic of preference is not a *purely comparative* or *strictly ordinal* one based

[13] Our simplified approach, for example, fails entirely to distinguish different degrees of favorableness and unfavorableness. But, of course, to the extent that such distinctions are drawn we move away from the aspirations of our simplicistic critic back toward our initial starting point.

solely upon the conception of preferability as such, but an *evaluative* one in which preference relations are based derivatively upon an essentially *quantitative* approach, the assessment (measure) of the intrinsic merit (goodness) of the objects involved. We treat preference as being *derivative* from merit assessments and not as an ultimately self-contained comparisory. To illustrate the distinction at issue, suppose that we are to deal with four possible worlds w_1–w_4, and suppose that we knew them to be so listed in *order of preference*. If this purely comparative preference-information were all that we knew, we would be wholly unable to say (1) by *how much* w_1 (say) is preferred to w_2, let alone (2) what *the intrinsic merit* (goodness/badness) of w_1 (say) might be. This purely comparative basis would prove insufficient for the quality-assessment (and thus essentially quantitative) processes that underlie the preferability-comparison of our semantics.

4. *Relations Between the Two Modes of Preference*

There is an interesting kinship between the two types of preference we have distinguished. Consider again the four possible worlds w_1–w_4 of the preceding section. Notice that

$$pP^{\#}q \text{ becomes } \frac{a+b}{2} > \frac{a+c}{2} \text{ or } b > c$$

and that

$$pP^{\star}q \text{ becomes } \frac{a+b}{2} - \frac{c+d}{2} > \frac{a+c}{2} - \frac{b+d}{2} \text{ or } b > c.$$

This suggests that #-preference and ★-preference are equivalent. But it is readily seen that this is not the case. For our procedure and our tabulations have to this point been based on the supposition that the variables involved – 'p', 'q', etc. – represent *independent* propositions: propositions devoid of logical interconnections of such a kind that the #-value assigned to one must have a bearing upon that assigned to the other(s). *Only under this independence presupposition that* 'p', 'q', etc. represent *independent* propositions do *#-preference and ★-preference come to coincide.* When this presupposition is not satisfied, the equivalence no longer obtains, as is shown by the following example. Consider:

$$pP(p \vee q).$$

First, let P represent #-preference, and let us go again to the four w_1 of the preceding section. Then

$$pP^{\#}(p \vee q)$$

will represent

$$\frac{a+b}{2} > \frac{a+b+c}{3} \text{ or } a+b>c.$$

If, on the other hand, P represents $\star$-preference, then

$$pP^{\star}(p \vee q)$$

will represent

$$\frac{a+b}{2} - \frac{c+d}{2} > \frac{a+b+c}{3} - d \text{ or } a+b+3d>5c$$

and it is perfectly clear that these two inequalities are not equivalent. Thus it will only be in the special case of independent relata (essentially, those which do not share a common variable) that the two modes of preference will coincide.

5. *The von Wrightean Semantics*

We turn now to a variant approach to the semantics of preference-logic which is designed to codify the approach of G. H. von Wright's recent monograph. We again suppose as starting point a series of possible worlds (state descriptions):

$$w_1, w_2, \ldots, w_n.$$

We suppose that the propositions at issue are generated by truth-functional compoundings of the w_i. We suppose further a 'ground floor' preference ordering of the w_i, allowing the possibility of indifference, say, for example:

$$w_4 > w_3 \simeq w_5 > w_1 > w_2.$$

(In such a list every possible world can occur just once, and a well ordering must result when $\simeq$-connected entries are identified.)

Let us now construe

$$\alpha P^{w} \beta$$

to mean:

> For every γ (independent of α and β)[14] we have it that every possible
>
> world w_i in which $\left\{\begin{matrix}\alpha \text{ true} \\ \beta \text{ false} \\ \gamma \text{ true}\end{matrix}\right\}$ is $>$-preferable to all the
>
> corresponding possible world(s) w_j in which $\left\{\begin{matrix}\alpha \text{ false} \\ \beta \text{ true} \\ \gamma \text{ true}\end{matrix}\right\}$.

Note that the condition on γ here plays the role of a requirement of 'other things being equal'.

This specification of a semantical interpretation of the P^w-relationship corresponds closely to the system of von Wright's theory. It accords entirely with the motivations and explanations of his discussion, and is such that all of his 'basic principles' prove acceptable.

Let us illustrate the workings of this von Wrightean semantics in a numerical rather than merely comparative setting. Consider the following eight possible worlds:

World	p	q	r	$\#(w_i)$
w_1	+	+	+	x_1
w_2	+	+	−	x_2
w_3	+	−	+	x_3
w_4	+	−	−	x_4
w_5	−	+	+	x_5
w_6	−	+	−	x_6
w_7	−	−	+	x_7
w_8	−	−	−	x_8

Note now that on the von Wrightean semantics we have:

$$pP^wq \text{ iff } x_3 > x_5 \text{ and } x_4 > x_6.$$

On the $P^\star$-semantics, on the other hand, we have:

$$pP^\star q \text{ iff } x_3 + x_4 > x_5 + x_6.$$

---

[14] Actually, in realistic applications of this machinery, one would want to require here not merely logical independence alone, but causal independence as well.

It is thus obvious that P^w preferability entails $P^\star$ preferability, but not conversely, so that the former is significantly more restrictive than the latter. (Indeed we shall shortly argue that there is good reason to think it to be actually too restrictive.) Again, by way of application of this machinery, note that '$(p \supset q)\ P^w p$' is a perfectly possible preference-situation, which would prevail under the circumstance that both $[x_1, x_5, x_7] > x_3$ and $[x_2, x_6, x_8] > x_4$.

Let us now develop the argument as to the restrictiveness of P^w. Let the case with the possible worlds be as above, and let it be supposed that:

$$x_3 = +1000$$
$$x_4 = \quad 0$$
$$x_5 = -1000$$
$$x_6 = +.0001.$$

Surely, in any intuitively plausible sense of the term, p is now 'preferable' to q (for when p is true we may gain as much as 1000 but can lose no more than 0, whereas when q is true we may lose as much as 1000, but can gain no more than .0001). But on the (overly safe) construction of the von Wrightean semantics we cannot say that p is preferable to q because in one case (among potentially countless ones) we may lose a bit more by p than by q.

Despite the interpretative shortcoming brought out in such examples, the von Wrightean semantic does, however, enjoy one important systematic advantage. It proceeds simply and solely on the basis of an ordinal preference ordering of the possible worlds, and does not call – as do the $P^\star$ and $P^\sharp$ relations, for an actual cardinal valuation of them. But actually this advantage is more seeming than real, since an ordering can always be transformed into a valuation by such devices as letting every possible world score 1 point for every other one that it excels in the rank ordering.

6. *Preference-Tautologies*

On the basis of the semantical machinery developed in Sect. 3 above, we are able to introduce the concept of a *preference-tautology*. Consider a preference-principle of the type:

$$pPq \rightarrow \sim (qPp)$$
$$pPq \rightarrow \sim qP \sim p$$
$$(pPq \,\&\, qPr) \rightarrow pPr.$$

Such a principle will be a $P^{\#}$-tautology (or a $P^{\star}$-tautology, respectively) if, when P is interpreted throughout as $P^{\#}$ (or $P^{\star}$, respectively), the principle goes over into a truth – *i.e.*, an arithmetical truth – with respect to every possible assignment of #-values to the possible worlds generated out of truth-combinations of the variables that are involved.

For example, to see that the second principle of the preceding list is a #-tautology, we consider the #-value assignment:

Possible world	#-value
w_1: $p \& q$	a
w_2: $p \& \sim q$	b
w_3: $\sim p \& q$	c
w_4: $\sim p \& \sim q$	d

Now '$pP^{\#}q \rightarrow \sim qP^{\#} \sim p$' amounts to

$$\frac{a+b}{2} > \frac{a+c}{2} \rightarrow \frac{b+d}{2} > \frac{c+d}{2}$$

that is, to

$$b > c \rightarrow b > c$$

which is an arithmetical truth.

On the other hand, it can be seen that the principle

$$pP^{\#}q \rightarrow \sim qP^{\#} \sim p$$

breaks down under substitution for the variables involved. For if we substitute '$p \vee q$' for 'q' we obtain

$$pP^{\#}(p \vee q) \rightarrow \sim (p \vee q) P \sim p$$

which amounts to

$$\frac{a+b}{2} > \frac{a+b+c}{3} \rightarrow d > \frac{c+d}{2}$$

that is, to

$$a + b > 2c \rightarrow d > 2c$$

which is obviously falsifiable.

It is thus crucially important to distinguish between *unrestricted* preference-tautologies such as

$$pP^{\#}q \rightarrow \sim (qP^{\#}p)$$

which – as the reader can check – proves acceptable under *any and every* substitution of the variables involved, and *restricted* preference-tautologies like

$$pP^{\sharp}q \rightarrow \sim qP^{\sharp} \sim p$$

which has unacceptable substitution instances. On the other hand it is readily shown that

$$pP^{\star}q \rightarrow \sim qP^{\star} \sim p$$

is unrestrictedly acceptable.

It is an interesting fact, inherent in their 'restricted equivalence', that despite their very great *conceptual* difference (*i.e.*, the very different meanings that attach to them), essentially the same preference theses obtain for $P^{\sharp}$ and $P^{\star}$: the only differences that can arise between them are those growing out of substitution restrictions. The sorts of preference theses that can bring out on the side of formal acceptability the conceptual difference between the two concepts will be those that turn on substitution-restrictions, such as

$$pP(p \vee q) \rightarrow \sim (p \vee q)\, Pp$$

which is acceptable for $P^{\star}$ but not for $P^{\sharp}$.

The possession of a semantically viable concept of a *preference tautology* is of the utmost importance from the logical point of view. For with its guidance, the question of the axiomatization of preference logic can meaningfully be raised and fruitfully dealt with. Our interests here falling on the semantical rather than the formal/axiomatic side, we shall not pursue this prospect further on the present occasion.

7. *Restricted and Unrestricted Quantification*

To provide ourselves with a systematic formal mechanism for recording the (for our purposes) pivotal distinction between two different modes of quantifications, we shall introduce the unrestricted propositional quantifier $\forall$ with

$$(\forall p)(\text{---}p\text{---})$$

to be construed as asserting that '---p---' holds with respect to *any and every* substitution for 'p', and the restricted propositional quantifier A with

$$(\mathsf{A}p)(\text{---}p\text{---})$$

to be construed as asserting (only) that '---p---' holds for all those substitutions for 'p' which do not involve other variables that occur in '---p---'.

Thus, for example, we would be in a position to assert the principle:

$$(\forall p)(\forall q)(pP^{\#}q \rightarrow \sim [qP^{\#}p]).$$

On the other hand it would not be correct to assert the principle,

$$(\forall p)\,(\forall q)\,(pP^{\#}q \rightarrow \sim qP^{\#} \sim p)$$

although it would, by contrast, be correct to assert the principle,

$$(\mathsf{A}p)\,(\mathsf{A}q)\,(pP^{\#}q \rightarrow \sim qP^{\#} \sim p).$$

The possibility is (of course) not to be excluded that in certain cases a mixture of these quantifiers is appropriate, so that one could assert a principle of the form

$$(\mathsf{A}p)\,(\forall q)\,(\text{---}p, q\text{---})$$

claiming, in effect, that 'p'-substitutions must be restricted, although 'q'-substitutions can be made unrestrictedly.

The ideas and procedures at issue here are applied and illustrated in the Appendix.

4. AN EXAMINATION OF SOME PREFERENCE PRINCIPLES

The most extensive, and doubtless the best-known treatment of preference logic is that of G.H. von Wright's book on *The Logic of Preference* (Edinburgh, 1963). Some brief suggestions are offered in ch. II of R.M. Martin's book *Intension and Decision* (Englewood Cliffs, N.J., 1963). A suggestive discussion can also be found in an article by R.M. Chisholm and E. Sosa, 'On the Logic of "Intrinsically Better"' (*American Philosophical Quarterly*, 3, 1966). On the semantical side, these writers all proceed on the basis of intuitive, unformalized considerations. It is thus of interest to examine their preference-principles from the angle of their $P^{\#}$ and $P^{\star}$-tautologousness. The results of such an examination are tabulated below:

VON WRIGHT	$P^{\#}$	$P^{\star}$
(W1) $pPq \rightarrow \sim (qPp)$	+	+

		$P^\sharp$	$P^\star$
(W2)	$(pPq \,\&\, qPr) \rightarrow pPr$[15]	+	+
(W3)	$pPq \rightarrow (p \,\&\sim q)\, P(\sim p \,\&\, q)$	+	+
	$(p \,\&\sim q)\, P(\sim p \,\&\, q) \rightarrow pPq$	+	+
(W4)	$[\sim(\sim p \,\&\sim q)\, P \sim(\sim r \,\&\sim s)] \rightarrow [(p \,\&\sim r \,\&\sim s)\, P$ $(\sim p \,\&\sim q \,\&\, r) \,\&\, (p \,\&\sim r \,\&\sim s)\, P(\sim p \,\&\sim q \,\&\, s) \,\&$ $(q \,\&\sim r \,\&\sim s)\, P(\sim p \,\&\, q \,\&\, r) \,\&\, (q \,\&\sim r \,\&\sim s)$ $P(\sim p \,\&\sim q \,\&\, s)]$	−	−
	$[(p \,\&\sim r \,\&\sim s)\, P(\sim p \,\&\sim q \,\&\, r) \,\&\, (p \,\&\sim r \,\&\sim s)\, P$ $(\sim p \,\&\sim q \,\&\, s) \,\&\, (q \,\&\sim r \,\&\sim s)\, P(\sim p \,\&\sim q \,\&\, r) \,\&$ $(q \,\&\sim r \,\&\sim s)\, P(\sim p \,\&\sim q \,\&\, s)] \rightarrow [\sim(\sim p \,\&\sim q)$ $P \sim(\sim r \,\&\sim s)]$	−	−
(W5)	$pPq \rightarrow [(p \,\&\, r)\, P(q \,\&\, r) \,\&\, (p \,\&\sim r)\, P(q \,\&\sim r)]$	−	−
	$[(p \,\&\, r)\, P(q \,\&\, r) \,\&\, (p \,\&\sim r)\, P(q \,\&\sim r)] \rightarrow pPq$	$(+)^1$	$(+)^2$

CHISHOLM-SOSA		$P^\sharp$	$P^\star$
(A1)	$= (W1)$	+	+
(A2)	$[\sim pPq \,\&\sim (qPr)] \rightarrow \sim (pPr)$	+	+
(A3)	$[\sim(pP \sim p) \,\&\sim(\sim pPp) \,\&\sim(qP \sim q) \,\&\sim(\sim qPq)] \rightarrow$ $[\sim(pPq) \,\&\sim(qPp)]$	+	+
(A4)	$[\sim(qP \sim q) \,\&\sim(\sim qPq) \,\&\, pPq] \rightarrow pP \sim p$	+	+
(A5)	$[\sim(qP \sim q) \,\&\sim(\sim qPq) \,\&\, qP \sim p] \rightarrow pP \sim p$	+	+

KEY

− unacceptable
\+ unrestrictly acceptable
(+) restrictedly acceptable

ANNOTATIONS
1. The appropriate quantifier-prefix is: $(\forall p)(\forall q)(\mathsf{A}r)$
2. The appropriate quantifier-prefix is: $(\mathsf{A}p)(\mathsf{A}q)(\mathsf{A}r)$

R. M. Martin (*op. cit.*) accepts inter alia two principles whose status is as follows:

	$P^\sharp$	$P^\star$
(M1) $(pPr \vee qPr) \rightarrow (p \vee q)\, Pr$	−	−
(M2) $pP(q \vee r) \rightarrow [pPq \,\&\, pPr]$	−	−

[15] (W1) and (W2) represent the *antisymmetry* and the *transitivity* of the preference relation. (Between them they entail irreflexivity, viz.,: $\sim(pPp)$.) These are the minimal rules for 'preference' classically insisted upon in *all* treatments of the subject by logicians, economists, etc.

Moreover, the converses of these two principles also fail to obtain for both of our modes of preference:[16]

	$P^{\sharp}$	$P^{\star}$
(M3) $(p \vee q)\,Pr \rightarrow (pPr \vee qPr)$	–	–
(M4) $(pPq \,\&\, pPr) \rightarrow pP(q \vee r)$	–	–

The same goes for various cognate rules, as may be seen from the tabulation:

THE STATUS OF VARIOUS PREFERENCE PRINCIPLES

Preference Principle	Von Wright	Chisholm Sosa	Martin	$P^{\sharp}$	$P^{\star}$	P^{w}
1. $pPq \rightarrow \sim(qPp)$	√	√	√	+	+	+
2. $(pPq \,\&\, qPr) \rightarrow pPr$	√	√	√	+	+	+
3. $pPq \rightarrow \sim qP \sim p$		×	√	(+)[1]	+	+
4. $\sim qP \sim p \rightarrow pPq$		×	√	(+)[1]	+	+
5. $pPq \rightarrow (p \,\&\, \sim q)\,P\,(\sim p \,\&\, q)$	√	×		+	+	+
6. $(p \,\&\, \sim q)\,P\,(\sim p \,\&\, q) \rightarrow pPq$	√	×		+	+	+
7. $[\sim(pP \sim p) \,\&\, \sim(\sim pPp) \,\&\, \sim(qP \sim q) \,\&\, \sim(\sim qPq)] \rightarrow [\sim(pPq) \,\&\, \sim(qPp)]$	√	√		+	+	+
8. $[\sim(qP \sim q) \,\&\, \sim(\sim qPq) \,\&\, pPq] \rightarrow pP \sim p$		√		+	+	+
9. $[\sim(qP \sim q) \,\&\, \sim(\sim qPq) \,\&\, qP \sim p] \rightarrow pP \sim p$		√		+	+	–
10. $pPq \rightarrow [(p \,\&\, r)\,P\,(q \,\&\, r) \,\&\, (p \,\&\, \sim r)\,P\,(q \,\&\, \sim r)]$	√			–	–	+
11. $[(p \,\&\, r)\,P\,(q \,\&\, r) \,\&\, (p \,\&\, \sim r)\,P\,(q \,\&\, \sim r)] \rightarrow pPq$	√			(+)[2]	(+)[3]	+
12. $[\sim(pPq) \,\&\, \sim(qPr)] \rightarrow \sim(pPr)$		√		+	+	–
13. $(pPr \vee qPr) \rightarrow (p \vee q)\,Pr$			√	–	–	–
14. $(p \vee q)\,Pr \rightarrow [pPr \,\&\, qPr]$	√			–	–	–
15. $[pPr \,\&\, qPr] \rightarrow (p \vee q)\,Pr$	√			–	–	–
16. $(p \vee q)\,Pr \rightarrow (pPr \vee qPr)$			√	–	–	–
17. $pP(q \vee r) \rightarrow (pPq \,\&\, pPr)$	√			–	–	–
18. $(pPq \,\&\, pPr) \rightarrow pP(q \vee r)$	√			–	–	–
19. $(pPr \,\&\, qPr) \rightarrow (p \,\&\, q)\,Pr$				–	–	–
20. $(p \,\&\, q)\,Pr \rightarrow (pPr \,\&\, qPr)$				–	–	–
21. $pP(q \,\&\, r) \rightarrow (pPq \,\&\, pPr)$				–	–	–
22. $(pPq \,\&\, pPr) \rightarrow pP(q \,\&\, r)$				–	–	–

[16] Note, however, that this specific feature of these rules does not exclude them from a proper and positive role in the logic of preference viewed in a wider perspective. Cf. the discussion in Sect. 5 below.

Preference Principle	Von Wright	Chisholm Sosa	Martin	$P\sharp$	$P\star$	P^{w}
23. $[\sim(\sim p \,\&\, \sim q)\, P \sim(\sim r \,\&\, \sim s)] \rightarrow$ $[(p \,\&\, \sim r \,\&\, \sim s)\, P\, (\sim p \,\&\, \sim q \,\&\, r) \,\&$ $(p \,\&\, \sim r \,\&\, \sim s)\, P\, (\sim p \,\&\, \sim q \,\&\, s) \,\&$ $(q \,\&\, \sim r \,\&\, \sim s)\, P\, (\sim p \,\&\, \sim q \,\&\, r) \,\&$ $(q \,\&\, \sim r \,\&\, \sim s)\, P\, (\sim p \,\&\, \sim q \,\&\, s)]$				−	−	+
24. $[(p \,\&\, \sim r \,\&\, \sim s)\, P(\sim p \,\&\, \sim q \,\&\, r) \,\&$ $(p \,\&\, \sim r \,\&\, \sim s)\, P(\sim p \,\&\, \sim q \,\&\, s) \,\&$ $(q \,\&\, \sim r \,\&\, \sim s)\, P(\sim p \,\&\, \sim q \,\&\, r) \,\&$ $(q \,\&\, \sim r \,\&\, \sim s)\, P(\sim p \,\&\, \sim q \,\&\, s)] \rightarrow$ $[\sim(\sim p \,\&\, \sim q)\, P \sim(\sim r \,\&\, \sim s)]$	√			−	−	−
	√					

KEY:

− unacceptable
\+ unrestrictedly acceptable
(+) restrictedly acceptable
√ explicitly accepted
× explicitly rejected

[1] The appropriate quantifier-prefix is: $(\mathsf{A}p)(\mathsf{A}q)$.
[2] The appropriate quantifier-prefix is: $(\forall p)\, \mathsf{A}q)(\mathsf{A}r)$.
[3] The appropriate quantifier-prefix is: $(\mathsf{A}p)\, (\mathsf{A}q)\, \mathsf{A}r)$.

Several features of this tabulation warrant comment:

(i) It is noteworthy that the only really uncontested principles are numbers (1) and (2) (i.e., irreflexivity and transitivity).[17]

(ii) It is striking that the various authorities are so seriously at odds with one another after going beyond the just-indicated point of common departure.

(iii) It is interesting that so few of the plausible-seeming principles listed after number (12) are acceptable on any of the three accounts of the matter here under consideration.

* * *

All this, I believe, goes far toward showing undesirability of proceeding by intuition in the construction of an axiomatic theory for the rules of preference-logic. The advantages of the semantical approach come strikingly to the fore.

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[17] In this regard it deserves remark how matters fare with the preference measure $P\S$ based on the valuation of a proposition in terms of a probabilistically weighted mean of the possible worlds in which this proposition is true. (With $P\sharp$ all these weights are set equal. Cf. footnote 10 above.) Here principles (1) and (2) survive, but even such plausible principles as (3)–(6), acceptable for all the other modes of P-preference, will fail to hold.

We now have a guide to the selection of preference-principles which safeguards us against the often paradoxical features of the deliverances of intuition. At the purely informal, intuitive level of understanding, a concept may well prove to be equivocal. In this case, it takes one form for which certain principles 'obviously' hold, and also a second form for which other, equally 'obvious' principles hold that are inconsistent with the former. The semantical approach protects us against this logically intolerable situation in which incompatible results confront us with equal plausibility.

Taking the semantical approach we can say that 'we know what we are doing' in a far more thoroughgoing way than is possible with any axiomatic treatment. Although serious problems doubtless still remain to be resolved, there can be little doubt that the *semantical* – in contrast to the *axiomatic* – approach affords the most promising prospects for the development of the logic of preference, and that the best hopes for future progress in this field lie in this direction.

* * *

Chisholm and Sosa discuss some principles of the logic of preference which have been accepted by certain writers, but which they themselves reject. These principles include the following:

		$P^{\#}$	$P^{\star}$
(R1)	$pPq \rightarrow \sim qP \sim p$	(+)	+
	$\sim qP \sim p \rightarrow pPq$	(+)	+
(R2) = (W3) above		+	+

But all of the preference principles rejected by Chisholm and Sosa are both $P^{\#}$ and $P^{\star}$-tautologies. The reasons for this divergence warrant brief consideration.

For specificity, let us focus attention upon (R1), with respect to which Chisholm and Sosa argue as follows:

> ... although the state of affairs consisting of there being happy egrets (p) is better than that one consisting of there being stones (q), the state of affairs that consists of there being no stones ($\sim q$) is not better, or worse, than that state of affairs consisting of there being no happy egrets ($\sim p$).

As this quotation brings out, Chisholm and Sosa do not deal with our

(completely characterized) possible worlds, but with particular states of affairs (i.e., with p's and q's rather than $(p \,\&\, q)$'s and $(p \,\&\, \sim q)$'s). Moreover, they proceed on the basis of what might be called the 'raw' or intrinsic propositional valuations, of the sort of which the following is a good example:

If it is the case that	then the resultant utility-value is
p	+ 4.0 units
$\sim p$	0.0 units
q	+ 0.5 units
$\sim q$	− 1.0 units

And here it is certainly true, with respect to such 'raw' valuations, that p's (first-order) preferability to q by no means guarantees not-q's preferability to not-p. But this fact does not conflict with our findings, which proceed on a quite different plane. For note that on our approach we would first transform the raw-valuations of the preceding scheme into valuations of alternative possible worlds:

Possible world	♯-value
$w_1: p \,\&\, q$	+ 4.5 units
$w_2: p \,\&\, \sim q$	+ 3.0 units
$w_3: \sim p \,\&\, q$	+ 0.5 units
$w_4: \sim p \,\&\, \sim q$	− 1.0 units

And the propositional ♯-valuations we would *then* derive – and thus the preferences that would be based upon them – would have a quite different structure, to wit:

Proposition	♯-value
p	+ 3.75 units
$\sim p$	− 0.25 units
q	+ 2.50 units
$\sim q$	+ 1.00 units

And on *this* basis of assessment it would have to be the case that – as we have shown – p's preferability to q guarantees not-q's preferability to not-p. In summary, the intuitive ideas operative in the Chisholm-Sosa concept of 'intrinsic preferability' in no way conflict or involve incompatibilities with the procedures and results of our formal semantics.

Exactly the same line of analysis applies to the Chisholm-Sosa line of objection to von Wright's (W3):

$$pPq \leftrightarrow (p \& \sim q)\, P(\sim p \& q).$$

Their counter-example is of the type:

If it is the case that	then one is to get
p	$+3$
$\sim p$	0
q	$+1$
$\sim q$	-2

Clearly one prefers p's happening to q's (*i.e.*, prefers a 3-unit gain to a 1-unit gain). But one certainly does not prefer $(p \& \sim q)$'s happening (when one gets $+1$) to $(\sim p \& q)$'s happening (when one also gets exactly $+1$). But let us translate this 'raw' valuation into our technical $\sharp$-valuation *via* the consideration of the possible worlds:

Possible world	$\sharp$-value
$w_1 : p \& q$	$+4$
$w_2 : p \& \sim q$	$+1$
$w_3 : \sim p \& q$	$+1$
$w_4 : \sim p \& \sim q$	-2

And now with respect to the derivative $\sharp$-values, it is clear that we could not have

$$pP^{\sharp}q \textit{ i.e.,} \frac{\sharp(w_1) + \sharp(w_2)}{2} > \frac{\sharp(w_1) + \sharp(w_3)}{2} \textit{ i.e.,} \sharp(w_2) > \sharp(w_3)$$

without also concurrently having:

$$(p \& \sim q)\, P^{\sharp}(\sim p \& q) \textit{ i.e.,} \sharp(w_2) > \sharp(w_3).$$

The proscription of contradiction-generating substitutions with respect to the $\sharp$-measure requires further discussion. Consider, for example, the principle (acceptable both for $P^{\sharp}$ and $P^{\star}$),

(W3) $\quad pPq \rightarrow (p \& \sim q)\, P(\sim p \& q)$

and let it be assumed that $\vdash p \rightarrow q$, so that we have

(W4) $\quad pPq \rightarrow cP(\sim p \& q) \quad c =$ a contradiction.

Now take a concrete example, letting

p = Having \$12 (*i.e.*, having at least \$12, that is, having \$12 or more)

q = Having \$11 (*i.e.*, having at least \$11, that is, having \$11 or more).

Note that *this p is preferable to this q* on any presystematic understanding of the matter, and that, moreover, p entails q. Consequently, we would come to be committed by W4 to:

cP (Having exactly \$11).

This consequence is clearly absurd. But it is not, in fact, a valid consequence of our logic of preference because one of the essential steps by which it was obtained involved a fallacious process of inference, to wit, a contradiction-generating substitution for the #-measure.

It will be objected that the principle at issue (W3) holds not only for #-preference, where we have insisted on excluding contradiction-generating substitutions, but also for ★-preference, where this restriction has been dropped. Consequently, so goes the objection, the indicated way out is not available. This objection is correct, so far as it goes, but it fails to realize that, because of the *technical* character of ★-preference, the entire difficulty at issue does not arise.

Let it be supposed, for the sake of simplicity, that \$20 is the maximum amount which, as a matter of the 'practical politics' of the situation, is at issue (nothing would be affected if this were fixed at \$100 or \$1,000). Then we shall have it that:

$$\#(p) \quad = \#(\$12 \text{ or more}) = \frac{12 + 13 + \cdots + 20}{9} = 16.0$$

$$\#(q) \quad = \#(\$11 \text{ or more}) = \frac{11 + 12 + \cdots + 20}{10} = 15.5$$

$$\#(\sim p) = \#(\$11 \text{ or less}) = \frac{0 + 1 + 2 + \cdots + 11}{12} = 5.5$$

$$\#(\sim q) = \#(\$10 \text{ or less}) = \frac{0 + 1 + 2 + \cdots + 10}{10} = 5.0.$$

As a consequence:

$$\star(p) = 16.0 - 5.5 = 10.5$$
$$\star(q) = 15.5 - 5.0 = 10.5.$$

It is thus simply not the case with respect to the technical concept of ★-preference now at issue that p ('Having \$12 or more') is preferable to q ('Having \$11 or more'). The difficulty at issue falls to the ground because one of its essential premisses fails to be true.

5. A MEASURE-THEORETIC PERSPECTIVE UPON THE LOGIC OF PREFERENCE

It is useful to look at our quantitative approach to the logic of preference from a somewhat different perspective. Let it be supposed that we have a family of propositions represented by the meta-variables 'α', 'β', 'γ', etc. These proportions are assumed to range over the set S, assumed to be closed under the familiar truth-functional connectives. Let there be a real-value measure μ with

$$\mu(\alpha)$$

defined over the set S, subject to the stipulation that equivalent propositions obtain the same μ-value, i.e., that:

$$\text{If } \vdash \alpha \leftrightarrow \beta, \text{ then } \mu(\alpha) = \mu(\beta).$$

Moreover, let it be supposed that our μ-measure is such as to satisfy the following additional condition:

$$\mu(\sim \alpha) = -\mu(\alpha).$$

(It should be observed that our measure $\bigstar(\alpha)$ is, whereas $\sharp(\alpha)$ is not, of such a kind as to meet this last-named condition.) It may be remarked, moreover, that this condition has the consequence that:

$$\text{If } \mu(\alpha) = \mu(\sim \alpha), \text{ then } \mu(\alpha) = 0.$$

Now it is readily verified that if we stipulate a preference-relation P^{μ} in such a way that

$$\alpha P^{\mu} \beta \quad \text{iff} \quad \mu(\alpha) > \mu(\beta)$$

then wherever the μ-measure satisfies the aforementioned conditions, then P^{μ} must satisfy all of the Chisholm-Sosa axioms (as well as the first two von Wright axioms). It is also readily verified that, under the stipulated conditions, we must have it that P^{μ} must satisfy the Chisholm-Sosa rejected thesis:

$$\alpha P^{\mu} \beta \leftrightarrow \sim \beta P^{\mu} \sim \alpha.$$

This way of approaching the matter at once systematizes our previous group of findings with respect to $P^{\bigstar}$.

* * *

Important consequences immediately follow when one assumes that the states of affairs corresponding to propositions make their value-contributions *independently* – i.e., that for the propositions at issue one has it that:

(I) $\mu(\alpha \& \beta) = \mu(\alpha) + \mu(\beta)$.

For consider two principles that have figured centrally in our discussions, viz., the *negativity* principle,

(N) $\mu(\sim \alpha) = -\mu(\alpha)$

and the *averaging* principle,

(A) $$\mu(\alpha) = \frac{\mu(\alpha \& \beta) + \mu(\alpha \& \sim \beta)}{2}.$$

It is readily shown, by elementary calculations, that:

If (I), then (N) iff (A).

That is, once (I) is given, the negativity principle (N) and the averaging principle (A) will come to the same thing.

It is also of interest to observe that once one has the negativity principle

$$\mu(\sim \alpha) = -\mu(\alpha)$$

there is no longer much point to introducing the ★-type counterpart measure for μ:

$$\mu^{\star}(\alpha) = \mu(\alpha) - \mu(\sim \alpha).$$

For then

$$\mu^{\star}(\alpha) = 2\mu(\alpha)$$

so that $\mu^{\star}$ is simply a linear transformation of μ, and not a genuinely new measure at all. The purpose of introducing a ★-type measure is as a normalizing step to obtain a negativity-principle; it is pointless when one already has a negativity principle.

* * *

It is of interest to re-examine, in the light of our generalized measure-theoretic approach, some of the principles previously found unacceptable

for $P^{\sharp}$ and $P^{\star}$. By way of example, let us return to the axiom (in the style of R. M. Martin)

(M2) $\quad pP^{\mu}(q \vee r) \rightarrow pP^{\mu}q \,\&\, pP^{\mu}r.$

This now becomes

$$\mu(p) > \mu(p \vee q) \rightarrow [(\mu(p) > \mu(q)) \,\&\, (\mu(p) > u(r))]$$

or equivalently:

$$\mu(p \vee q) \geqslant \max[\mu(p), \mu(r)].$$

Thus any μ-measure that is a monotonically increasing function of its Boolean constituent will satisfy (M2). This condition, while unquestionably plausible for certain propositional measures (e.g., probability), is patently unsuitable for a measure of 'goodness'.

Again, consider the axiom:

(M1) $\quad (pP^{\mu}r \vee qP^{\mu}r) > (p \vee q)P^{\mu}r.$

This now becomes

$$[(\mu(p) > \mu(r)) \vee (\mu(q) > \mu(r))] \rightarrow \mu(p \vee q) > \mu(r)$$

or equivalently

$$\mu(p \vee q) \geqslant \min[\mu(p), \mu(q)].$$

This, of course, would also be guaranteed immediately by the previous condition.

6. CONCLUSION

It has been the main aim of this chapter to provide a systematically developed semantical theory for the logic of preference. Using as starting point the orthodox semantical notion of a 'possible world' we have adopted the idea of a valuation-measure for such worlds as a determinant of preferabilities. This apparatus has been applied to appraise the acceptability of various preference-principles accepted on the basis of informal considerations by the several writers who have to date attempted to systematize the logic of preference. Our method has to some extent been able to reconcile the divergent approaches proposed in the literature. But in any case,

sufficient evidence has, I trust, been provided to indicate the power and promise of the suggested line of approach.[18]

7. APPENDIX. RESTRICTED VS. UNRESTRICTED QUANTIFICATION

When we consider the tautologousness of a preference-principle such as

$$pPq \rightarrow \sim qP \sim p$$

for a specific construction of P, such as $P^{\#}$ or $P^{\star}$, we must – as we saw in Sect. 3 above – inquire into the style of universal quantifier that is to prevail over the variables involved. The purpose of this appendix is to explain and illustrate the sort of checking procedure involved in the type of tautology-testing that is at issue here.

Let us first consider the $P^{\#}$ interpretation of P:

(I) $\quad pP^{\#}q \rightarrow \sim qP^{\#} \sim p.$

And let us begin by understanding this principle as asserted with respect to weak (i.e., restricted) quantification:

(Ia) $\quad (\mathsf{A}p)\,(\mathsf{A}q)\,[pP^{\#}q \rightarrow \sim qP^{\#} \sim p].$

Consider now an (arbitrary) index of merit measure # for the relevant possible worlds w_i, as follows:

Possible worlds	#-values
$w_1: p \,\&\, q$	a
$w_2: p \,\&\, \sim q$	b
$w_3: \sim p \,\&\, q$	c
$w_4: \sim p \,\&\, \sim q$	d

Given our canonical interpretation of $P^{\#}$, (I) is now rendered as a relationship between arithmetical inequalities as follows:

$$\left(\frac{a+b}{2} > \frac{a+c}{2}\right) \rightarrow \left(\frac{b+d}{2} > \frac{c+d}{2}\right)$$

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[18] This chapter is based on the author's paper on 'Semantic Foundations fort he Logic of Preference' in N. Rescher (ed.), *The Logic of Decision and Action* (Pittsburgh, 1967), pp. 37–62. I wish to acknowledge the help of John Robison and Anne Cross (Mrs. Michael) Pelon in working out some of the ideas of the paper.

or equivalently

$$(b > c) \rightarrow (b > c).$$

The acceptability of this, its arithmetical transform, at once establishes the acceptability of (I) when construed as (Ia).

Let us next consider whether we can strengthen this to

(Ib) $\quad (\forall p)\,(\mathsf{A}q)\,[pP^{\#}q \rightarrow \sim qP^{\#} \sim p]$

that is – ask whether we can make arbitrary substitutions for 'p', putting for 'p' also formulas involving 'q'. We must in particular examine the result of putting in place of 'p' such replacements as:

$$q, \sim q, p \,\&\, q, p \,\&\, \sim q, p \vee q, p \vee \sim q.$$

(In fact, this list must prove sufficient.) But now, when we put '$p \,\&\, q$' for 'p' in (I), we obtain

(II) $\quad pP^{\#}(p \,\&\, q) \rightarrow \sim (p \,\&\, q)\, P^{\#} \sim p$

whose arithmetical transform is

$$\left(\frac{a+b}{2} > a\right) \rightarrow \left(\frac{b+c+d}{3} > \frac{c+d}{2}\right).$$

Since this is not a truth of arithmetic, we see that (II) is not a #-tautology, and therefore leave it that (Ia) cannot be strengthened to (Ib).

But – on the other hand – can (Ia) be strengthened to:

(Ic) $\quad (\mathsf{A}p)\,(\forall q)\,[pP^{\#}q \rightarrow \sim qP^{\#} \sim p]$?

Again let us examine the result of 'q'-substitutions. Consider putting '$p \vee q$' for 'q':

(III) $\quad pP^{\#}(p \vee q) \rightarrow \sim (p \vee q)\, P^{\#} \sim p.$

The arithmetical transform of this $P^{\#}$-principle is

$$\left(\frac{a+b}{2} > \frac{a+b+c}{3}\right) \rightarrow d > \frac{c+d}{2}$$

which is clearly not a truth of arithmetic. Thus (III) is not a $P^{\#}$-tautology, and consequently we cannot strengthen (Ia) to (Ic). In the face of these

findings, it now goes without saying that (Ia) cannot be strengthened to:

(Id) $(\forall p)(\forall q)[pP^{\#}q \rightarrow \sim qP^{\#} \sim p]$.

Let us turn now to the $P^{\star}$-interpretation of our initial preference-principle:

(IV) $pP^{\star}q \rightarrow \sim qP^{\star} \sim p$.

Again, let us begin by understanding this principle as asserted with respect to weak (i.e., restricted) quantification:

(IVa) $(\mathsf{A}p)(\mathsf{A}q)[pP^{\star}q \rightarrow \sim qP^{\star} \sim p)$.

This version of the principle yields the arithmetical transform

$$[\star(p) > \star(q)] \rightarrow [\star(\sim q) > \star(\sim p)]$$

which, by the definition of the $\star$-measure, amounts to

$$([\#(p) - \#(\sim p)] > [\#(q) - \# \sim (q)]) \rightarrow ([\#(\sim q) - \#(q)] > [\#(\sim p) - \#(p)])$$

which, being of the form

$$(x > y) \rightarrow (-y > -x)$$

is a truth of arithmetic.

It remains to be seen whether (IVa) can actually be strengthened to:

(IVb) $(\forall p)(\forall q)[pP^{\star}q \rightarrow \sim qP^{\star} \sim p]$.

Let us try the effect of some particular substitution, say '$p \,\&\, q$' for 'p'. Then (IVb) yields

(V) $(p \,\&\, q)\, P^{\star}q \rightarrow \sim qP^{\star} \sim (p \,\&\, q)$

whose arithmetical transform is

$$\left[\left(a - \frac{b+c+d}{3}\right) > \left(\frac{a+c}{2} - \frac{b+d}{2}\right)\right] \rightarrow \left[\left(\frac{b+d}{2} - \frac{a+c}{2}\right) > \left(\frac{b+c+d}{3} - a\right)\right]$$

which, being of the form

$$(x > y) \rightarrow (-y > -x)$$

is again a truth of arithmetic. So far so good. And the general fact that all substitution instances of

$$pP^{\star}q \rightarrow \sim qP^{\star} \sim p$$

are $p^{\star}$-tautologies is readily established – as follows: Regardless of the substitutions made in (IV), the result will take the form

$$\alpha P^{\star}\beta \rightarrow \sim \beta P^{\star} \sim \alpha$$

whose arithmetical transform will be

$$([\#(\alpha) - \#(\sim \alpha)] > [\#(\beta) - \#(\sim \beta)]) \rightarrow$$

$$([\#(\sim \beta) - (\#\beta)] > [\#(\sim \alpha) - \#(\alpha)])$$

which is readily seen – on analogy with the preceding – to be a truth of arithmetic.

It is thus clear that the preference-principle we have selected for examination,

$$pPq \rightarrow \sim qP \sim p$$

is unrestrictedly tautologous for $P^{\star}$, but is only a restricted tautology for $P^{\#}$.

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CHAPTER XVI

DEONTIC LOGIC

1. THE DEONTIC MODALITIES

Those propositional modalities which deal with such normative conceptions as the permitted, the obligatory, or the forbidden, are characterized as *deontic* modalities. The extensive investigations of the logical theory of these concepts which have been carried on in recent days have been stimulated by an important paper published by G.H. von Wright in 1951.[1]

Deontic logic deals with the permitted, obligatory, etc., in a highly abstracted way. As a branch of logical theory, it leaves to substantive disciplines – such as ethics and law – the concrete question of exactly what acts or states of affairs are to be forbidden, permitted, or the like. It seeks to systematize the abstract, purely conceptual relations obtaining between propositions in this sphere, such as the following:

> If an act is to be obligatory, then it must be permitted. In given circumstances, any specified act is either permitted itself or else its omission is permitted.

A system of deontic logic to be considered here is based upon a series of conditional (dyadic or two-place) propositional modalities, as follows:

> P(p/q) for 'p is *permitted* given q'
> O(p/q) for 'p is *obligatory* given q'
> F(p/q) for 'p is *forbidden* given q'.

These are to be interpreted along the lines of construing P(p/q) to say: 'the action of realizing p – i.e., of so acting as to render it true that p is the case as the result, or partial result, of one's actions – is permitted in

[1] 'Deontic Logic', *Mind* **60** (1951) 1–15. For a general bibliography of the subject see A. R. Anderson 'The Formal Analysis of Normative Systems', in N. Rescher (ed.) *The Logic of Decision and Action* (Pittsburgh, 1967).

circumstances in which it is true that q'. These three deontic modalities are interdefinable; for example, we need only take P as primitive and then define O and F as follows:

$$F(p/q) \text{ for } \sim P(p/q)$$
$$O(p/q) \text{ for } \sim P(\sim p/q).$$

Thus p is forbidden (given q) if it is not the case that p is permitted (given q), and p is obligatory (given q) if $\sim p$ is forbidden (given q).[2]

Since our interest here relates to the formal logical theory of these conceptions – rather than to the details of their concrete application – we shall abstract wholly from the question of the nature or warrant of the permission (obligation, i.e.,) at issue. That is, we shall waive the question of whether it is *ethically* permitted, or *legally* permitted, or permitted by rules of etiquette, or permitted in the context of some other body of rules of action, such as the rules of a game. Thus some examples of the sorts of propositions we would symbolize in terms of $F(p/q)$ are:

1. Taking a (or this) thing is forbidden to someone who is not authorized to take it.
2. Castling is forbidden to someone whose king is in check.
3. Heavy drinking is forbidden to someone who has to drive a car shortly.

2. THE PROBLEM OF FOUNDATIONS

To construct the formal logical machinery for handling these deontic modalities, an appropriate semantical basis must be developed. Let us again begin with the idea of possible worlds familiar from modal logic (and already used in the preceding chapter).

Let us again suppose that we have a stock of particular, specific, concrete propositions, say $p_1, p_2, p_3, \ldots$. For the sake of simplicity let it be supposed

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[2] Not every deontic relationship is definable in these terms, however. Consider

M (p/q) for 'p is *meritorious* gives q'

i.e., that when q it is good (and so, of course, permitted), but not necessarily obligatory, to bring it about that p. In terms of our machinery we cannot define this particularly interesting deontic idea, whose features we shall not, however examine further here.

that there are just four of them p_1–p_4. Given these propositions, we can again make an enumeration of various alternative possible worlds that can be specified with regard to them, these worlds which vary with the different assumptions one might make about the truth-status of the basic propositions. A *possible world* is thus given by some particular specification of the truth-situation for all of the propositions at issue. In this way, a set of possible worlds is presented by a tabulation of the following sort, a tabulation in which each column indicates (by an entry of T or F), with respect to all the propositions at issue, whether they are to be true or false in the possible world corresponding to a given row:

	p_1	p_2	p_3	p_4
W_1	T	T	F	F
W_2	T	F	F	T
W_3	T	T	F	T
W_4	T	F	F	F
W_5	F	F	F	T

For any basic (elementary atomic) proposition p_i; such a truth-table determines a set of the W_i in which p_i is true. For any complex proposition p that is built up out of the elementary p_i's by the truth-functional connectives ($\&, \vee, \sim, \supset$, etc.), we can specify the set $\{p\}$ such that:

$\{p\}$ = the set of all the possible worlds W_i where p is true.

One can work out what is the set at issue by the usual truth-table methods. In this way we can determine the truth-sets { } for compound propositions, using essentially the rules of calculation.

$\{\sim p\} = \{p\}'$, i.e., the set-complement of $\{p\}$ in the set $W = \{W_1, W_2, W_3, \ldots\}$

$\{p \,\&\, q\} = \{p\} \cap \{q\}$

$\{p \vee q\} = \{p\} \cup \{q\}$.

Moreover, it is clear that for virtually any sort of implication-relation, if p implies q, then $\{p\} \subseteq \{q\}$, that is, whenever p is true, then q must also be true.

We can now introduce – in a way closely analogous with this situation as regards *truth* – a comparable sort of analysis with respect to permission. The starting-point here is a permission-table constructed on the following lines:

	p_1	p_2	p_3	p_4
W_1	Y	Y	Y	N
W_2	Y	Y	N	N
W_3	N	N	Y	N
W_4	Y	N	Y	N
W_5	N	Y	Y	N

Here an entry of Y = *yes* – for example, that in the very first place – means that the realization of the corresponding p_i is permitted[3] in the corresponding W_j (and, in the example, p_1 is permitted in that world in which actually p_1 is true, p_2 is true, p_3 false, and p_4 false. Similarly, an entry of N = *no* means that realization of the corresponding act-proposition is not permitted to its appropriate agents in the corresponding possible world.

Such a permission-table of course only fixes the permission-status of the elementary (atomic) act-proportions. The question of the status of compound proportions remains to be settled. This issue is a problematic one. If p and q are both permitted in a possible world, then $p \& q$ need not be permitted – or indeed need not even be possible – in it (since p and $\sim p$ may both be permitted in a possible world).

On analogy with the set $\{p\}$ of possible worlds in which p is true, let us define a set $[p]$ as follows:

$[p]$ = the set of possible worlds in which p is *permitted.*

Our question regarding the permission states of compound propositions comes down to this: What are the rules of calculation for [ ] when propositions involving connectives are involved?

It seems plausible – indeed pretty well inevitable – that one should have

$[p \vee \sim p] = \mathrm{V}$ (Where $\mathrm{V} = W$ is the entire universe of possible worlds)

Since of course one must have

$[p] \cup [\sim p] = \mathrm{V}$ (i.e., $= W$)

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[3] That is, permitted to those agents who are involved in its realization.

we thus have it that:

$$[p \vee \sim p] = [p] \cup [\sim p].^4$$

But we cannot go beyond this point to establish any stronger general relationship between $[p]$ and $[\sim p]$. It being clear, for example, that these two sets are by no means mutually exclusive: the doing and non-doing (omission) of some act may well be permitted in one and the same possible world. For [ ], unlike { }, there is thus no simple calculation-rule for negation: knowing the set $[p]$ of possible worlds in which p is permitted, we can say nothing in a formal, general way about $[\sim p]$, the set of possible worlds in which $\sim p$ is permitted – apart from this, that $[\sim p]$ must include (at least) all of those possible worlds not included in $[p]$.

Before considering the situation with regard to other connectives such as disjunction and conjunction, it is useful to settle, by way of preliminary, an importantly related issue – namely that of implication. It is plain that for *any* type of implication relation (IMP) we must have it that:

$$\text{If } p \text{ IMP } q, \text{ then } \{p\} \subseteq \{q\}.$$

We will certainly not, on the other hand, have an analogous rule for [ ]. This is shown by the so-called *Good-Samaritan Paradox*. First consider the statements:

(1) X helps the victim of an unjustifiable attack

(2) Someone is (has been) the victim of an unjustifiable attack.

Clearly (1) implies (2) while – equally clearly – possible worlds in which (1) is permitted (for example, our own) need not by any means be ones in which (2) is permitted. To avoid such paradoxes, we need a particularly strong form of implication, namely entailment or deducibility on logical grounds alone. Symbolizing this relationship by $\rightarrow$, we shall require this strong mode of entailment as basis for the principle:

$$\text{If } p \rightarrow q, \text{ then } [p] \subseteq [q].$$

But we shall in general reject the weaker form:

$$\text{If } p \text{ IMP } q, \text{ then } [p] \subseteq [q].$$

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[4] We cannot – as will shortly be seen – generalize this to:

$$[p \vee q] = [p] \cup [q].$$

For untoward consequences arise here when the mode of IMP is among the weaker modes of logical relationship such as the presuppositional type of link at issue in the *Good-Samaritan Paradox.*[5]

We have it for the strong mode of implication (namely entailment) that:

$$\text{If } p \rightarrow q, \text{ then } [p] \subseteq [q].$$

We shall designate this as *the entailment principle.* According to this principle, we at once obtain the rules of calculation:

$$[p \,\&\, q] \subseteq ([p] \cap [q]), \text{ but not necessarily conversely;}^{6}$$
$$([p] \cup [q]) \subseteq [p \vee q].^{7}$$

This pair of rules – distinctly poorer than those for { }, for which both converses also hold – provide us with as much as we can get by any of rules of calculation for conjunction and disjunction for [ ]. We cannot strengthen the relationship because, in effect, of the 'interference problem' posed by the fact that each of two acts might be permitted separately but not conjointly.

It follows from the above, that one cannot specify the permission status of a set of propositions in the same economical way that one can settle the question of their truth status. For when *permission* is at issue then it will not serve, as it will with *truth*, to indicate the situation as regards a handful of basic propositions and leave all the rest to the rules of calculation. In the specification of permission status one must be explicit, since one does not have a shortcut of this sort at one's disposal.

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[5] Note that the statement (1) and (2) of the paradox can be construed so that the linkage is one of deducibility (and not some form of 'presupposition'), for example:

(1a) $(\exists x)(\exists y)[Hxy \,\&\, \mathrm{V}y]$
(1b) $(\exists y)\,\mathrm{V}y.$

But if construed thus – so that (1) does indeed entail (2), it is no longer permissible for X (or anyone) to bring about (1) since this requires bringing it about that $(\exists y)\,\mathrm{V}y$.

[6] Consider, for example, a possible world in which p and q both are each permitted provided that the other is not done. This world will be an element of $[p] \cap [q]$, but not of $[p \,\&\, q]$, and so the preceding inclusion is not reciprocal.

[7] There appears to be no decisive reason for not postulating the converse of this rule, and thus have:

$$([p] \cup [q]) = [p \vee q].$$

3. TWO CONSTRUCTIONS OF CONDITIONAL PERMISSION

We are now in a position to set up a logical system for the deontic modalities on the basis of the idea of permission-sets as set forth in the last section. Taking $P(p/q)$ as the basic relationship, we propose to explore the consequences of defining this in two alternative ways:

(1) $P_1(p/q)$ if (and only if) p is permitted in *all* of the cases (possible worlds) in which q is true, that is, if every possible world in which q is true is also one in which p is permitted – in short, when $\{q\} \subseteq [p]$.

(2) $P_2(p/q)$ if (and only if) p is permitted in most of the cases (possible worlds) in which q is true, that is, if the majority (more than half) of the possible worlds in which q is true are ones in which p is permitted – in short, when $\{q\} \ll [p]$ (where $A \ll B$ when the number of A's failing to be B's is less then that of A's that are B's).

In other words, we have $P_1(p/q)$ if *p is always permitted whenever q is true* and $P_2(p/q)$ if p is *generally permitted when q is true.*

Thus for example, let the situation with respect to truth and permission be indicated for a family of possible worlds as follows:

	p	q	(1) Perm(p)?	(2) Perm($\sim p$)?	(3) Perm(q)?	(4) Perm($\sim q$)?	(5) Perm(p & q)?	(6) Perm($p \vee q$)?
W_1	T	T	yes	yes	yes	yes	yes[b]	yes[d]
W_2	T	F	no	yes[a]	no	yes	no	no
W_3	F	T	yes	no	no	yes	no[c]	yes
W_4	F	F	no	yes	no	yes	no	no

NOTES

(a) Here the 'no' of column (1) forces a 'yes' entry. Note that we can never have 'no' in both columns (1) and (2) or in both columns (3) and (4) for any row.

(b) Note that in virtue of the 'yes' of columns (1) and (2), the entailment principle permits – but does not require – a 'yes' entry here; whereas a 'yes' entry here *forces* a 'yes' entry in columns (1) and (3).

(c) Note that the entailment principle would forbid our writing 'yes' here because of the 'no' in column (3).

(d) Note that the entailment principle forces a 'yes' entry here because of the 'yes' in column (1) – and/or that in column (3).

Given the data assembled in this tabulation, it is readily checked that the following results will obtain:

Permission Claim	Truth Status	Permission Claim	Truth Status
$P_1(p/p)$	false	$P_1(p \& q/p \vee q)$	false
$P_2(p/p)$	false	$P_2(p \& q/p \vee q)$	false
$P_1(p/q)$	true	$P_1(p \vee q/p \& q)$	true
$P_2(p/q)$	true	$P_2(p \vee q/p \& q)$	true
$P_1(q/p)$	false	$P_1(p/p \& q)$	true
$P_2(q/p)$	false	$P_2(p/p \& q)$	true
$P_1(p \& q/p)$	false	$P_1(p/p \vee q)$	false
$P_2(p \& q/p)$	true	$P_1(p/p \vee q)$	true
$P_1(p \& q/q)$	false		
$P_2(p \& q/q)$	true		

This listing illustrates the *modus operandi* of the two proposed constructions of the conception of conditional permission.

It is particularly illuminating for our two alternative constructions to examine in the light of them the principle:

$$P(p/q) \vee P(\sim p/q).$$

Suppose the situation with respect to truth and permission for a family of possible worlds to be as follows:

	p	q	Perm(p)?	Perm($\sim p$)?
W_1	T	T	yes	no
W_2	T	F	no	yes
W_3	F	T	no	yes
W_4	F	F	yes	no

Note now that we do not have the principle:

$$P_1(p/q) \vee P_1(\sim p/q).$$

Nor does the analogous principle hold for P_2. The failure of the two systems to endorse this principle represents a fact of particular importance. It is the systematic reflection of the fact that the P-relation of our systems is inherently general rather than particular in nature: it construes its relata

in a generic rather than concrete manner. For it is obvious that in any *particular* circumstance q – and in any *particular* possible world W – an act of a given sort p must either be permitted or not. But of course what $\mathrm{P}(p/q)$ says with respect to our constructions is that p is *always* or *mostly* permitted when q is given.

4. THE LOGIC OF CONDITIONAL PERMISSION

Given these two specifications for the intended meaning of the relationship of conditional permission, the systematization of the logical rules of a deontic theory can be accomplished along the familiar semantical lines. The criterion of acceptability of deontic principles is simply this, that a principle is acceptable (i.e., valid) if it is true for every possible interpretation of its constituent variables – i.e., if it is true no matter how we specify the 'possible worlds' at issue.

Let us now apply these ideas to the testing of certain proposed theses of deontic logic. Consider the thesis:

$$\mathrm{P}(p \vee \sim p/q).$$

Taking P in the sense of P_1, we may ask whether $\mathrm{P}_1(p \vee \sim p/q)$ holds, i.e., whether:

$$\{q\} \subseteq [p \vee \sim p].$$

Since we have it that $[p \vee \sim p] = V$, this is necessarily true. Again, taking P in the sense of P_2 we ask whether $\mathrm{P}_2(p \vee \sim p/q)$ holds, that is, whether:

$$\{q\} \ll [p \vee \sim p]?$$

But the previous $\subseteq$-relationship immediately guarantees this $\ll$-relationship also.

To take a somewhat more complex case, consider the thesis:

$$(p \rightarrow q) \rightarrow (\mathrm{P}(p/r) \rightarrow \mathrm{P}(q/r)).$$

Given construction P_1, this comes to amount to:

$$(p \rightarrow q) \rightarrow ((\{r\} \subseteq [p]) \rightarrow (\{r\} \subseteq [q])).$$

But since, in view of the entailment principle, we must have

$$(p \rightarrow q) \rightarrow ([p] \subseteq [q])$$

this result at once follows. And the argument for P_2 goes through in exactly the same way. For $A \subseteq B$ guarantees that:

$$(C \ll A) \rightarrow (C \ll B).$$

To obtain yet another example, let us consider the thesis:

$$P(p \,\&\, q/r) \rightarrow P(p/q \,\&\, r).$$

Let us first check the validity of this thesis for construction P_1. This question comes to:

$$\{r\} \subseteq [p \,\&\, q] \rightarrow \{q \,\&\, r\} \subseteq [p]?$$

Noting that, since $(p \,\&\, q) \rightarrow p$, the entailment principle leads us to $[p \,\&\, q] \subseteq [p]$, and that in general $\{r \,\&\, q\} \subseteq \{r\}$, we see that the question at issue must be answered in the affirmative.

Again, let us check the validity of our thesis for interpretation P_2. Here the question to be resolved is

$$\{r\} \ll [p \,\&\, q] \rightarrow \{q \,\&\, r\} \ll [p]?$$

Now assume $\{r\} \ll [p \,\&\, q]$. Then since $[p \,\&\, q] \subseteq [p]$ by the implication principle, we have $\{r\} \ll [p]$. But now the fact that $\{q \,\&\, r\} \subseteq \{r\}$ does not lead to the desired result – that is, $\{q \,\&\, r\} \ll [p]$ – since we cannot get from the premisses $A \subseteq B$ and $B \ll C$ to the conclusion $A \ll C$. The thesis at issue thus does not hold for P_2, although it does hold for P_1.

As our final example, let us test the acceptability with respect to P_1 of the thesis:

$$P(p \vee q/r) \rightarrow [P(p/r) \vee P(q/r)].$$

This question comes to that of the acceptability of:

$$\{r\} \subseteq [p \vee q] \rightarrow ((\{r\} \subseteq [p]) \vee (\{r\} \subseteq [q])).$$

Assuming (as above) that $[p \vee q] = ([p] \cup [q])$, this comes down to the question of whether:

$$C \subseteq (A \cup B) \rightarrow ((C \subseteq A) \vee (C \subseteq B))$$

which must of course be answered in the negative. Thus the thesis in question – unlike its converse! – is not acceptable for P_1. Nor does the thesis hold for P_2, since

$$C \ll (A \cup B) \rightarrow ((C \ll A) \vee (C \ll B))$$

also fails. On the other hand, it is again readily seen that the converse will obtain.

Having illustrated the procedure for a validity-check of deontic theses in the foregoing examples, we simply report the situation with respect to these and several other theses in summary form:

Thesis	Validity Status P_1	Validity Status P_2
$P(p \vee \sim p/q)$	$\surd$	$\surd$
$P(p \vee q/r) \rightarrow (P(p/r) \vee P(q/r))$	$\times$	$\times$
$(P(p/r) \vee P(q/r)) \rightarrow P(p \vee q/r)$	$\surd^*$	$\surd^*$
$(p \rightarrow q) \rightarrow (P(p/r) \rightarrow P(q/r))$	$\surd^*$	$\surd^*$
$P(p \,\&\, q/r) \rightarrow P(q/r)$	$\surd^*$	$\surd^*$
$P(p \,\&\, q/r) \rightarrow P(p/q \,\&\, r)$	$\surd^*$	$\times$
$(P(p/q) \,\&\, P(q/r)) \rightarrow P(p/r)$	$\times$	$\times$
$P(p/q \vee \sim q) \rightarrow P(p/r)$	$\surd$	$\times$
$P(p/q) \rightarrow P(p/r \,\&\, \sim r)$	$\surd$	$\times$
$P(p/q) \rightarrow P(p/q \,\&\, r)$	$\surd$	$\times$
$P(p/q) \vee P(\sim p/q)$	$\times$	$\times$

KEY:

$\times$ unacceptable (invalid)

$\surd$ acceptable (valid)

$\surd^*$ acceptable provided that the 'entailment principle' is espoused.

As this tabulation shows, our two different interpretations of the operator P of conditional permission generate markedly divergent theses for deontic logic. The two alternative constructions of meaning for this relationship result in radically distinct systems for the formalization of deontic logic.[8,9]

[8] The principal ideas and results of the present chapter are drawn from the author's paper on 'Semantic Foundations for Conditional Permission', *Philosophical Studies* **19** (1968). For other interesting applications of this line of approach see G. H. von Wright, 'Deontic Logics', *American Philosophical Quarterly* **4** (1967) 136–143.

[9] **(Added in proof.)** R. Z. Parks has suggested in correspondence that when one introduces a monadic, one-place permission operator **P** subject to the identity $\{Pp\} = [p]$, then '$P_1(p/q)$' will amount to '$q \supset Pp$', since: $P_1(p/q)$ iff $\{q\} \subseteq [p]$ iff $\{q\} \subseteq \{Pp\}$ iff $q \supset Pp$. One can thus reduce the conditional deontic modality represented by P_1 to an unconditional modality, leaving only the 'statistical', P_2-style relationship as a genuinely and irreducibly conditional modality.

CHAPTER XVII

DISCOURSE ON A METHOD

1. INTRODUCTION

This final chapter is a study in philosophical methodology, or rather, a study of one particular method of philosophical investigation: the *method of applied logic.* In its most general outline, this method consists in the application of the tools and concepts of modern formal logic to the study of philosophical questions. It is important to emphasize at the very outset that the method to be described is *not* being put forward as a general all-purpose procedure of philosophical investigation. There is no intent here to endorse any technique as *the* method of philosophical analysis. The present method is avowedly conceived as constituting a promising tool for the investigation of one class, and only one class of philosophical problems, namely those which hinge upon considerations regarding the *formal* and strictly logical aspects of a philosophical concept or proposition. Thus the method to be described is put forward *not* as a generic procedure of philosophical analysis, but only as a technique of limited applicability, suitable for the investigation of one restricted, albeit very important, class of philosophical problems.

No claim is made for the originality of the method which is here under discussion. It has been used repeatedly, often with what seems to me to be significant success, by many logicians since the turn of the present century.[1] However, I have tried here to characterize this method in an overt, explicit, and deliberate way. My aim is to lay the basis for a more intensive consideration of its rationale than is elsewhere available, at least to my knowledge. We have been concerned throughout the book with those branches of logic that are of substantial relevance to philosophical

[1] Some classic examples are: Hugh MacColl on existence (*Mind* **14** (1905) 74–81), Bertrand Russell on denoting (*Mind* **14** (1905) 479–493), C. I. Lewis on implication (*Mind* **21** (1912) 522–531), and G. E. Moore on external and internal relations (*Proceedings of the Aristotelian Society* **20** (1920) 40–62).

inquiry. It thus seems fitting to include some methodological observations as to how the formal machinery of symbolic logic is effectively to be employed in philosophical analysis.

The application of techniques of modern logic to the study of philosophical issues is not a matter of borrowing the finished end-products of one field for use in the investigation of problems in another field, as is frequently the case in applied mathematics. It is not in this sense of an employment of certain accomplished results that I speak of 'applied logic', but in the sense of borrowing from logic certain of its tools, i.e., concepts, formalizing techniques, methods of inference, etc. The keystone of the method is the concept of *formalization*: the construction of a formal framework whose concepts are more sharply defined and whose logical interrelationships are more explicitly articulated. Such a framework is not, of course, constructed *in vacuo*; it is deliberately designed to serve as a *counterpart* to some informal field of ideas which bears upon philosophical questions. Every feasible step is taken to assure that the formal system is as closely analogous as possible to its informal original. When such a construction has been carried through, and proper care has been taken to assure ourselves of its adequacy, the resultant system can be used as an apparatus for the study of questions arising within the context of our informal ideas regarding the subject matter in view.

The hope which motivates the formalizing synthesis is that the more powerful logical techniques of analysis and inference that are available in the context of formalized systems can be brought to bear helpfully upon questions of interest arising within the less manageable sphere of informal ideas. The gain in exactness and precision will – it is hoped – aid in diagnosing the exact sources of problems and difficulties, and afford instrumentalities for their resolution. But let us defer further discussion about its uses until a closer look has been taken at the method itself.

2. OUTLINE OF THE METHOD

The structure of the method here in question consists in the procedural sequence composed of the following five steps:

(I) On the basis of a careful survey of the domain to be investigated, our informal, 'intuitive' understanding of this field is

summarized and systematized in a set of *criteria of adequacy* for any formal theory capable of being reasonably regarded as an adequate formalization for the concepts of this specific domain.

(II) Guided by these criteria of adequacy, a provisional, tentative formulation of a formal theory is attempted by means of the machinery of symbolic logic.

(III) This formal theory is tested against all of the criteria of adequacy to assure that all of these explicit requirements upon any acceptable theory are satisfied by the proposed theory.

(IV) The principal logical consequences of the proposed theory are developed. A *de novo* check is made to assure that all of these also turn out to be acceptable from the standpoint of our informal, intuitive ideas regarding acceptability and adequacy for theories of the domain under investigation.

(V) The theory is now *applied* to an examination of some of the principal problems and to an analysis of some of the major conceptual issues and difficulties arising within the domain. In this way, an assessment is made of the capacity of the proposed theory to clarify the issues and to aid resolving the problems and perplexities of the field.

This method unites two of the styles of philosophical methodology that have been widely current in recent discussions. Step I involves the informal analysis characteristic of the 'ordinary language' school of philosophical analysis. Step II calls for that type of precise systematization characteristic of the 'rational reconstruction' approach.

Before considering more closely the epistemological character of the method outlined here, it is helpful to compare this method with the hypothetico-deductive method of scientific investigation, for it is clear at first sight that a close analogy obtains between them. Scientific method is commonly schematized under essentially the following rubrics:

(1) *Observation.* Through careful and systematic observation, the basic data regarding the phenomena of the domain of investigation are determined.

(2) *Hypothesis.* Based on these observational data, a hypothetical conjecture of a theory that fits the observed phenomena is made.

(3) *Verification.* The theory is tested by its employment in relation to new (and often contrived, i.e., experimental) circumstances and situations. First, the predictive consequences of the theory in such new situations are elicited by a process of logical deduction from the theory. Then it is verified that the actual observations in these cases accord with the expectations on the basis of inference from the theory.

(4) *Integration.* An assessment is made of the extent to which the theory: (1) 'fits' into the framework of other relevant theories that have already attained to the status of being well-established, (2) simplifies and systematizes the explanation of the phenomena falling within its scope, and (3) proves to be generally fruitful in terms of its consequences for the understanding of a significant domain of natural phenomena.

Comparison of these two methods of inquiry at once brings to light a close similarity of structure. For it is quite plain that step (I) is an analogue of (1) (getting the facts of the area of investigation into view), step (II) and (III) jointly comprise an analogue of (2) (devising a theory *prima facie* adequate to the facts), step III is an analogue of (3) (testing the theory by a check on the acceptability of its consequences), and finally, step (V) is an analogue of (4) (a determination of the usefulness of the theory). Thus the method of conceptual investigation that has been outlined here is, in general structure, a close analogue of the hypothetico-deductive method of empirical scientific investigation. Despite radical disparities with regard to the subject-matter areas to which these methods apply, their schematic structure is closely similar.

A number of the implications that are suggested by this analogy between the method of the natural sciences and the method here under discussion warrant explicit consideration.

3. EXPERIMENTAL NATURE OF THE METHOD

The first (quite correct) suggestion that derives from the analogy, is that

the results of an application of the method must of necessity be of a provisional and tentative status. It is a commonplace that a scientific theory never becomes established with final and absolute certainty, but is always susceptible of refutation or modification in the light of new data; a scientific theory is always at the mercy of new and refuting observations. Analogously, any system that is the product of application of the present method can be upset when it fails to accord with newly discovered 'facts', i.e., when it leads to hitherto unforeseen but intuitively unacceptable consequences. This possibility can never be excluded altogether: no matter how far the theory goes in according with our informal understanding of the domain of investigation, the possibility of a disagreement can never be written off altogether. It is an inevitable fact that our logical and conceptual foresight is never so keen that the contingency that a proposed conceptual systematization may lead to unacceptable consequences can be excluded from the realm of possibility.

On the method of inquiry here presented, each particular investigation is based on a formal systematization of our informal ideas regarding the domain in view. But a structure cannot be significantly sounder than the foundations upon which it rests. Some degree of looseness and infirmity in coherence, uniformity, and clarity is inherent in our *informal* ideas regarding any subject matter domain. And this looseness in the foundations must be reflected in a provisional standing of the resultant systematization.

Clearly, however, this provisional nature of its product must not be viewed as a *shortcoming* of the present method, any more than is the case with the scientific method. The charge of defectiveness would be able to take hold only if such provisionality, while inherent in *this particular method* of investigation, were evitable by other means. This, however, is clearly not so. Every method of inquiry regarding concepts of the sort here in question must ultimately rest on our informal intuitive ideas regarding these concepts, since it is these informal and familiar concepts that constitute the objects of investigation. Since no method of investigation can yield results that are firmer than its data, and since in a conceptual inquiry these data must inevitably be our informal (presystematic) conceptions, this feature of the present method can not reasonably be viewed as a shortcoming or defect.

This provisional status of its products is reflected in the 'experimental' nature of the method. A scientific theory can be tested (and thus be liable

to upset) *experimentally* in terms of its capacity to underwrite correct anticipations of new types of observational situations. Analogously, a conceptual theory can be tested, as it were 'experimentally' by its capacity to underwrite appropriate resolutions of conceptual obscurities or difficulties. This is observed in a perceptive remark of Bertrand Russell's:

> A logical theory may be tested by its capacity for dealing with puzzles, and it is a wholesome plan, in thinking about logic, to stock the mind with as many puzzles as possible, since these serve much the same purpose as is served by experiments in physical science.[2]

As regards the method of investigation here in question, this analogy of Russell's is fully correct. The 'experimental' test of a conceptual theory is precisely its capability to resolve outstanding puzzles and difficulties relating to the concepts in view.

4. ANALYTICAL CHARACTER OF THE METHOD

In an application of the method under discussion, the first and fundamental phase of the entire procedure is one of analysis. The development of suitable criteria of adequacy for an adequate theory of the concept under study requires a detailed analytical survey of the concept, including an examination of the main features of our informal ideas regarding the conceptual domain in question, and examination of the customary usage of its key words, and the like. Only by means of such a careful prior analysis of the concepts involved can the foundations be laid for construction of a theory regarding them.

In such an analysis, a threefold purpose is served. First, our informal ideas are surveyed, put into some systematic order, and consequently clarified. Secondly, the data for the constructive phase of a formal theoretic synthesis are secured. Finally, those areas which can stand to profit by a constructive systematization are determined, since it is seen just what are the scope and limits of the area which we can adequately oversee by informal means alone. The preliminary informal analysis is thus essential

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[2] Pp. 484–485 of 'On Denoting', *Mind* **14** (1905) 479–493.

for determining the boundaries of the informally unexplorable terrain which a constructive theory must cover, as well as for providing the materials from which the actual construction must set out.

5. SYNTHETICAL NATURE OF THE METHOD

The key step in an application of the foregoing method is, however, constructive and *synthetical* in nature. Here the prior analysis is used as a basis for the construction of a conceptual system or theory in which the logical articulation of the field of study is revealed. The analysis of our informal concepts is taken as basis for projecting a formal system whose logical structure is articulated with an exactitude and precision that exceeds greatly that of our informal ideas. Criteria of adequacy derived from our informal notions are taken as blueprints for the construction of a formal theory in which the principal features of the conceptual domain under study, as revealed by analysis of its informal characteristics, are regularized and systematized.

A formal theory constructed by means of the present method must not in any way come into outright conflict or contradiction with our informal, presystematic conceptions. If such a conflict did arise, the whole purpose of the inquiry would be defeated, for we would no longer be studying the concepts which we had set out to investigate, but radically different ones. This is why the method of counter-examples is such a useful and powerful instrument in these conceptual investigations. An analysis starting from the ordinary, informal concepts of a given domain must, if acceptably executed, end with results that are fully compatible with ordinary conceptions.

However, while the systematized conceptual reworking that constitutes a theory of the kind our method is designed to provide must not *conflict* with our informal conceptions, they may, and indeed should go beyond them in such important respects as precision of meaning and explicitness of logical relationships. Ernest Nagel has urged this point with characteristic cogency:

> No... system of formal logic is or can be just a faithful transcription of those inferential canons which are embodied in common discourse, though in the construction of these systems hints may (I would say *must* – N.R.) be taken from current usage; for the entire *raison d'être* for such systems is the need for

precision and inclusiveness where common discourse is vague and incomplete, even if as a consequence their adoption as regulative principles involves a modification of our inferential habits.[3]

The current popularity of analytical techniques of philosophical enquiry should not blind us to the usefulness and propriety of synthetical, i.e., system-constructive, techniques.

It might be urged in objection that, given the fact that our informal conceptions act as ultimate arbiter under the present method, there is no point in concerning ourselves with the 'different' concepts to be encountered in a formalized system based upon them. This objection can be formulated in the form of a dilemma. When the system *agrees* with our informal ideas, it is dispensable (since we could rely upon them alone), while when it *disagrees* with our intuitions, it evinces its own inadequacy, since such disagreement must be taken as evidence for its incorrectness and inacceptability. The dilemma rests on a mistake. It closes its eyes to the frail, partial, and fragmentary character of our informal ideas regarding such concepts as lie within the proper sphere of application of the present method. *Ex post facto* agreement of intuition with systematically attained results is a very different thing from direct intuitive attainment of the results themselves. Our informal conceptions prove, in such matters, to be only a very partial guide. They start us off in the proper direction, and they help us from time to time along the way, but they do not guide us along the whole course of the journey. Here, the task of systematization becomes indispensable. Only when these are properly collected, collated, focussed, systematized, and (judiciously) projected, can our informal ideas provide adequate guidance. And in this way positions can be reached that could never be attained by unassisted intuition acting directly. The best analogy, and one upon which I myself place great weight, is that of mental calculation *vis à vis* longhand computations. The systematic procedures (i.e., longhand computations) must never conflict or prove to be in any way incompatible – with the informal instrumentalities (mental calculation). But the formal machinery that comprises the systematic procedures vastly extends the range of our

[3] 'Logic Without Ontology' in Y. H. Krikorian (ed.), *Naturalism and the Human Spirit* (New York, 1944), reprinted in H. Feigl and W. Sellars, *Readings in Philosophical Analysis* (New York, 1949), pp. 191–210 (quoted from p. 205).

logical vision, and enables us to see our way clear to conclusions not within the reach of informal procedures alone.

6. CONCLUSION: JUSTIFICATION OF THE METHOD

I have dwelt at some length above on the similarity obtaining between use of the method of applied logic in philosophical analysis here under discussion and the hypothetico-deductive method of scientific inquiry. This has been done solely in order to clarify the nature of the former method through a comparison with one that is already familiar. I do *not* claim that this analogy in any way tends to justify or validate this method. Such justification of the method cannot derive from its resemblance to procedures and technique derived from other fields; it must come in quite another way.

The pragmatic theory of truth espoused by Peirce, James, Dewey, and others, takes practicability as its criterion of truth, assessing truth by such questions as 'Does it work?', 'Can it be applied and used successfully?', 'Is it viable in *practice*?'. It is apparent that this theory simply borrows its criterion of truth from the field of methodology, i.e., the *theory of method.* And however dubious and questionable the pragmatic thesis may be as a *criterion of the truth of propositions or theories*, it is clearly quite correct and wholly valid within its original and proper sphere: as a criterion of the acceptability and justification of *methods.* The vindication of a method (or tool, instrumentality, procedure, practice, and the like) can clearly be based on the extent to which it *works*, and upon this criterion alone. As a doctrine regarding the criterion of evaluation for methods (etc.) pragmatism is perfectly sound, however futile and erroneous it may be regarded as a doctrine as to the truth of propositions.

When these general considerations are applied to the specific instance of the method of philosophical analysis which I have been concerned to describe, we conclude that the question of its justification ought not to be raised as a distinct and isolated issue. For if a justification is to be had, it cannot be based upon any general or abstract considerations: it must be sought in the specific applications of the method. The sole justification that can be given to this (or any other method) is one based upon realization that it stands or falls with the success of such instances of its application: its justification will have to rest upon its success in

furnishing the means for solving specific, concrete problems. Numerous detailed investigations which can be found in a wide sector of the contemporary American and British philosophical literature will provide some samples or examples of the kinds of results in terms of which the validity or propriety of the method here under discussion will have to be evaluated. And this task is one which I must invite the reader to undertake for himself.[4]

[4] This chapter is a somewhat expanded version of an article of the same title published in *Methodos* **11** (1959) 81–89.

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